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CORPS OF ENGINEERS, U. S. ARMY

FROST INVESTIGATIONS

FISCAL YEAR 1953

FIRST INTERIM REPORT
ANALYTICAL STUDIES
OF
FREEZING AND THAWING OF SOILS

by

Harl P. Aldrich, Jr.

Henry M. Paynter

Contract No. DA-19-016-eng-2314



TECHNICAL REPORT NO. 42

UNDER CONTRACT
WITH

ARCTIC CONSTRUCTION AND FROST EFFECTS LABORATORY
NEW ENGLAND DIVISION
BOSTON, MASSACHUSETTS

FOR

OFFICE OF THE CHIEF OF ENGINEERS
AIRFIELDS BRANCH
ENGINEERING DIVISION
MILITARY CONSTRUCTION

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NOTATION

The nomenclature used in this report agrees generally with that recommended in "Soil Mechanics Nomenclature", ASCE Manual of Engineering Practice No. 22, 1941. A partial list of recommended symbols and new symbols is given below:

<u>Symbol</u>	<u>Description</u>	<u>Units</u>
α	Thermal Diffusivity	sq.ft. per hr.
C_s	Specific Heat of Soil Solids	BTU per lb.
C_f	Volumetric Heat Capacity of Frozen Soil	BTU per (cu.ft.)(°F)
C_u	Volumetric Heat Capacity of Unfrozen Soil	BTU per (cu.ft.)(°F)
F	Freezing Index	Degree Days
k_f	Thermal Conductivity of Frozen Soil	BTU per (ft.)(°F)(hr.)
k_u	Thermal Conductivity of Unfrozen Soil	BTU per (ft.)(°F)(hr.)
L	Latent Heat of Fusion of Soil Moisture	BTU per cu.ft.
n	Surface Transfer Coefficient	(Dimensionless)
q	Rate of Heat Flow	BTU per (sq.ft.)(hr.)
Δs	Finite Distance	ft.
t	Time (For frost penetration formulas and nomograph: Duration of Freezing Index)	hr. (days)
u	Internal Energy, total heat content	BTU per cu.ft.
v	Temperature	°F
v_o	Degrees F by which mean annual temperature exceeds freezing point of soil moisture	°F
w	Water Content of Soil	

<u>Symbol</u>	<u>Description</u>	<u>Units</u>
x	Depth of Frost Penetration	feet (or inches)
$\Delta x, \Delta y, \Delta z$	Finite Distances in the x, y, and z directions	ft.
α	Thermal Ratio	(Dimensionless)
β	$\equiv \frac{a \Delta t}{\Delta s^2}$	(Dimensionless)
λ	Correction Coefficient	(Dimensionless)
μ	Fusion Parameter	(Dimensionless)

SYNOPSIS

This report contains the results of studies related to the prediction of the frost penetration depth below ground surfaces. In addition, the so-called thaw-consolidation problem has been studied and the results are reported by the writers.

Part II is devoted to the presentation of a mathematical formulation of the problem derived from first principles of heat transfer. The fluid and electric analogies to thermal problems are treated in some detail since they are of especial value not only for visualizing the thermal problem but as important aids in solving complex situations. Furthermore, because a closed-form analytical solution is impossible except for the simplest cases, considerable effort has been devoted to a discussion of practical methods of computations such as numerical and machine solutions. Descriptions of an hydraulic model and an electronic analog computer applicable to freezing and thawing problems are presented.

The second section of Part II deals with the writers' rational formula for the depth of frost penetration. Results of statistical studies involving actual frost penetration data are described. The effects of cyclic variations in surface temperature and of multilayered systems on the depth of freezing are presented.

Part III deals with all phases of the writers' depth of frost penetration nomograph including instructions for its use and its limitations and assumptions. A copy of the nomograph is included on the inside of the back cover of this report.

The final part of this document treats specifically the thaw-consolidation problem which arises during the spring melting period and with which pavement weakening and possible failure are associated. The problem is still in the formulation stage. Field investigations of the thaw-consolidation process and an hydraulic analog apparatus are needed at this time to advance further our knowledge of the problem.

PREFACE

This report represents the results of the first stage of an investigation which has the following ultimate aims:

- a. Development of improved methods of computing depth and rate of freezing and thawing in soils.
- b. Determination of the relationship between the load-supporting capacity of soils during thawing and the pertinent affecting variables (herein called the "thaw-consolidation" problem), and the development of improved engineering design criteria therefrom.

Active work on the program was initiated by the Arctic Construction and Frost Effects Laboratory in May 1952. At that time general outlines of the needed studies were prepared. These outlines anticipated that the studies would be made in two distinct phases: (a) Theoretical and analytical investigations, to be carried out by contract, (b) Practical application of the results, through formulation of engineering design criteria, to be carried out by Arctic Construction and Frost Effects Laboratory personnel. The outline of the basic physical concept of the thaw-consolidation phenomenon, as given in the outline of desired work, was prepared by Mr. K. A. Linell, Chief, Arctic Construction and Frost Effects Laboratory, on basis of previous personal observation in the field.

After review of the initial Arctic Construction and Frost Effects Laboratory outlines, Drs. Harl P. Aldrich, Jr. and Henry M. Paynter of the Massachusetts Institute of Technology submitted a definite proposal presenting their proposed methods for making the desired theoretical and analytical studies. A contract was prepared, and the studies reported herein were carried out during Fiscal Year 1953.

The report itself, which was prepared by Drs. Aldrich and Paynter, is self-explanatory of their work under the contract. During the course of the studies, the contractors held frequent conferences with Mr. James F. Haley, Assistant Chief, Arctic Construction and Frost Effects Laboratory, at which agreements were reached from time to time concerning the direction of the continuing studies. Under direction of Mr. Haley, the Arctic Construction and Frost Effects Laboratory also conducted laboratory tests under controlled conditions to check the validity of the frost penetration formulae developed by Drs. Aldrich and Paynter and made statistical studies comparing the theoretical frost depths with depths determined from field measurements during previous frost investigational programs.

In the present report Drs. Aldrich and Paynter analyze in detail the penetration of freezing below the ground surface and present results of their initial studies of the thaw-consolidation problem. The investigations will be continued by Drs. Aldrich and Paynter in Fiscal Year 1954; it is planned to make a more comprehensive attack upon the thaw-consolidation problem and to extend the present study of freezing to a consideration of the methods of computing depth and rate of thawing.

PART I. INTRODUCTION

1-01. CONTRACT. -

On 23 June 1953, the writers entered into a one year contract with the Corps of Engineers, New England Division for the frost studies presented in the following pages. This is the final report covering all phases of that contract, No. DA-19-016-ENG-2314.

The following paragraphs, which are quoted from Appendix "A" of the contract, describe the investigations which the writers agreed to conduct:

1. "The Contractor shall prepare nomographic charts for use in estimating the depth of frost penetration beneath paved and unpaved surfaces, and shall make a comprehensive theoretical analysis of the supporting capacity of thawing soils to determine the relationship between rate of thaw, consolidation and strength.
2. During the first phase of the investigation, the Contractor shall review existing literature on mathematical methods of computing frost penetration with the objective of determining which of the various formulae is most suitable. The Contractor shall then prepare charts to permit the solution of this formula without the need of laborious computations and still include the following variables affecting the depth of freezing, air freezing index, moisture content of soil, thermal properties of soil (i.e., thermal conductivity, specific heat, and latent heat), freezing point of soil moisture and surface temperature transfer coefficient.
3. The Contractor shall investigate more accurate methods of analyses of frost penetration problem with a view to numerical and machine solutions of the resulting equations, giving due considerations to the effect of such

factors as surface insulation and soil stratification. During the second phase of the investigation the contractor shall, by means of theoretical mathematical analyses, attempt to develop practical equations and analyses to express the void ratio or saturated water content of any type of soil as a function of time during the frost melting period."

1-02. SCOPE. -

While the proposed scope of this investigation is outlined under Section 1-01, in actuality the scope has been considerably broader. Principal additions covered in this report are (1) the presentation of a new formula for predicting the depth of frost penetration and (2) results of statistical studies conducted to verify the formula. The SYNOPSIS serves to point out other factors considered within the scope of this investigation.

1-03. PERSONNEL. -

The contract studies and the preparation of this report were carried out primarily by the writers referred to as the Contractor in the contract. Several students at the Massachusetts Institute of Technology shared in conducting statistical analyses, performing computations, and preparing drawings for monthly progress reports and for the nomograph.

PART II. ANALYTICAL STUDIES

2-01. MATHEMATICAL FORMULATION. -

a. Synopsis. - This article treats the mathematical basis for studies of thermal diffusion, depth of frost and thaw penetration, as well as the seepage diffusion concepts required for analytical studies of the thaw-consolidation problem.

After stating the first principles from which all developments begin, both exact differential equations and approximate difference equations are given which embody these principles. Moreover, practical and fruitful fluid and electric analogies to the thermal problem are outlined.

In order to overcome the handicap to analysis imposed by the complex latent heat conditions, several practical machine and hand methods of computation are presented.

b. Fundamental Principles. - At any point and instant in a thermal medium, such as soil, the local rate of change of total heat content (or internal energy) u must exactly balance the net efflux of heat. This amounts to a statement that heat is conserved and in the absence of heat sources, its flow must satisfy the continuity equation:

$$\frac{\partial u}{\partial t} + \text{div } \vec{q} = 0 \quad \dots (2-1)$$

Moreover, this directed heat flow $\vec{q}$ is experimentally found to be proportional to the negative temperature gradient, with the factor of proportionality defined as the thermal conductivity k of the substance. This heat flow law thus yields the second fundamental equation:

$$\vec{q} = -k \vec{\text{grad}} \quad v \quad \dots (2 - 2)$$

where v is the instantaneous temperature field in the material.

These two principles were first set forth by J. B. J. Fourier in his classic treatise on the "Theorie Analytique de Chaleur" in 1822, and they form the basis of all rational investigations of heat conduction phenomena.

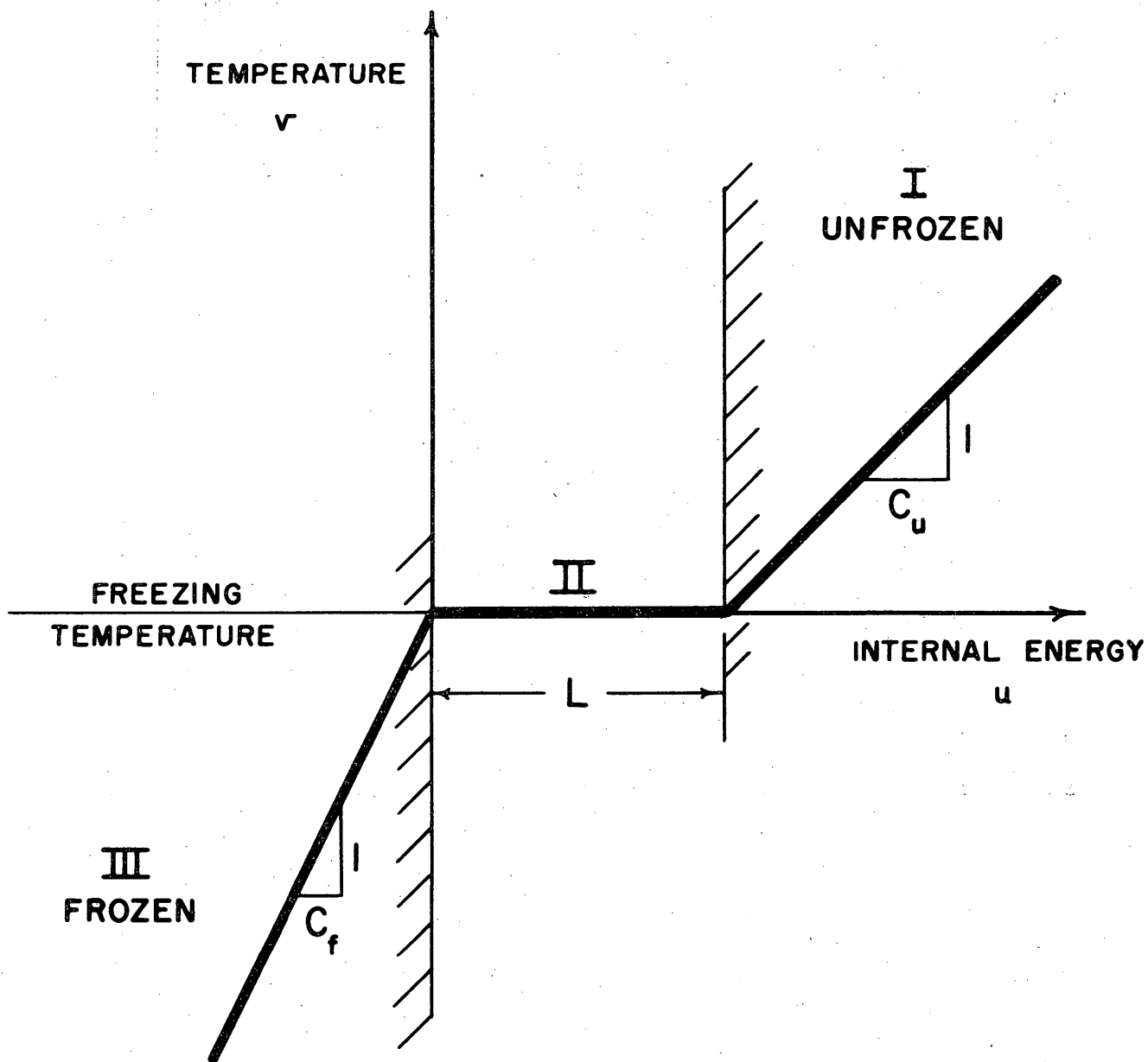
The heat content u of a substance is found experimentally to depend primarily upon its temperature v , and one finds under constant volume conditions, at least, that from a consideration of energy conservation, it must vary only with the temperature. For example, the experimental curve of the internal energy u of water as a function of temperature v is shown as Figure 1. Three distinct, effectively linear segments are distinguished, as marked by the numbers I, II and III.

The region I denotes the fluid state of water in which the temperature v increases approximately in proportion to the increase in stored heat u . This proportionality or slope we may define as the volumetric specific heat (under constant volume) C_u in the unfrozen or liquid phase, or:

$$C_u = \left[\frac{\partial u}{\partial v} \right]_{\text{unfrozen}} \quad \dots (2 - 3)$$

In the region marked III, the substance (water) is in the frozen or solid phase and the slope of the curve may be measured by the frozen volumetric specific heat C_f , where:

$$C_f = \left[\frac{\partial u}{\partial v} \right]_{\text{frozen}} \quad \dots (2 - 4)$$



INTERNAL ENERGY DIAGRAM FOR WATER

FIGURE I

The flat or horizontal portion II of the internal energy curve displays the heat content which must be released or absorbed to produce the phase change, and which is defined as the latent heat of fusion L of the material.

As stated, the curves I and III are not exactly straight, but the deviations are usually negligible such that the small errors introduced by assuming C_u and C_f both constant are consistent with the assumption of negligible unit volume change, upon change of phase.

Under these conditions, except in the region of the fusion (or freezing) temperature, the variation in u with time may be directly related to the variation in v , as the following:

$$\frac{\partial u}{\partial t} = C \frac{\partial v}{\partial t} \quad \dots(2 - 5)$$

where C becomes C_u or C_f depending on whether v is above or below freezing, respectively.

Substituting this last expression, together with Equation (2 - 2), into Equation (2 - 1) yields:

$$C \frac{\partial v}{\partial t} + \text{div} (-k \text{ grad } v) = 0 \quad \dots(2 - 6)$$

c. Differential Equations. - If the substance is homogeneous such that C and k are constant throughout a region, and is entirely either above or below the fusion temperature, then the following differential equation holds for the temperature field v :

$$C \frac{\partial v}{\partial t} = k \nabla^2 v \quad \dots(2 - 7)$$

This last expression may be re-arranged to give the thermal diffusion equation:

$$\frac{\partial v}{\partial t} = a \nabla^2 v \quad \dots(2 - 8)$$

in which $a = k/C$ is called the thermal diffusivity. For three-dimensional (x, y, z) problems, this equation would be written:

$$\frac{\partial v}{\partial t} = a \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) \quad \dots(2 - 8a)$$

For two-dimensional (x, y) situations it becomes:

$$\frac{\partial v}{\partial t} = a \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \quad \dots(2 - 8b)$$

while for one-dimensional (x) problems there remains:

$$\frac{\partial v}{\partial t} = a \frac{\partial^2 v}{\partial x^2} \quad \dots(2 - 8c)$$

d. Approximate Equations. - While the expressions in the previous sections describe the frost penetration problem in a mathematically correct fashion, exact solutions can be found only for a small number of idealized cases, due to the complex conditions of the latent heat transfer and other effects.

When such conditions arise in engineering problems, it is the customary practice to resort to largely empirical techniques to obtain reliable prediction formulas. However, a useful alternative follows from the substitution of an approximate system of simpler mathematical equations, which may either be solved directly in closed form, or which may be solved by various forms of computation or analogies.

In particular, it is often found effective to replace the continuous conduction equation (written in one-dimensional form for simplicity):

$$q = -k \frac{\partial v}{\partial x} \quad \dots(2-9)$$

by a "lumped" approximation for a finite layer n of thickness $\Delta x = d_n$, in the form:

$$q_n = -k_n \frac{(v_{n+1} - v_n)}{d_n} \quad \dots(2-10)$$

or, by defining a thermal resistance $R_n = d_n/k_n$, as:

$$q_n = + \frac{(v_n - v_{n+1})}{R_n} \quad \dots(2-11)$$

The thermal continuity equation is here considered to hold for lumps of material concentrated at the terminals or end points of a series of such resistors; the relation between the net heat input and the stored heat for the n -th such lump would then be given by the equation:

$$\frac{du_n}{dt} = q_{n-1} - q_n \quad \dots(2-12)$$

and the relation between the temperature v_n and heat u_n , is indicated by the form:

$$v_n = f_n(u_n) \quad \dots(2-13)$$

where the function f_n resembles the curve of Figure 1.

For hand computations and digital computing machines the continuous thermal continuity Equation (2-12) may be expressed discretely in terms of the finite difference approximation:

$$\frac{du_n}{dt} \approx \frac{u_n(t + \Delta t) - u_n(t)}{\Delta t} \quad \dots(2-14)$$

to give:

$$u_n(t + \Delta t) - u_n(t) = \Delta t [q_{n-1}(t) - q_n(t)] \quad \dots(2-15)$$

In terms of a three-dimensional space lattice, the differential Equation (2-8a) may be approximately represented by the finite difference equation:

$$\frac{v_{\Delta t} - v_0}{\Delta t} = a \left[\frac{v_1 + v_2 - 2v_0}{(\Delta x)^2} + \frac{v_3 + v_4 - 2v_0}{(\Delta y)^2} + \frac{v_5 + v_6 - 2v_0}{(\Delta z)^2} \right] \quad \dots(2-16)$$

where:

$$\begin{aligned} v_{\Delta t} &= v(x, y, z, t + \Delta t) \\ v_0 &= v(x, y, z, t) \\ v_1 &= v(x + \Delta x, y, z, t) \\ v_2 &= v(x - \Delta x, y, z, t) \\ v_3 &= v(x, y + \Delta y, z, t) \\ v_4 &= v(x, y - \Delta y, z, t) \\ v_5 &= v(x, y, z + \Delta z, t) \\ v_6 &= v(x, y, z - \Delta z, t) \end{aligned}$$

If $\Delta x = \Delta y = \Delta z = \Delta s$, this may be abbreviated to the form:

$$\frac{v_{\Delta t} - v_0}{\Delta t} = \left(\frac{a \Delta t}{\Delta s^2} \right) (v_1 + v_2 + v_3 + v_4 + v_5 + v_6 - 6v_0) \quad \dots(2-17)$$

or, with $\beta \equiv a \Delta t / \Delta s^2$, to the expression:

$$v_{\Delta t} = \beta \sum_{k=1}^6 v_k + (1-6\beta)v_0 \quad \dots(2-17a)$$

In the same way, Equation (2-8b) becomes:

$$v_{\Delta t} = \beta \sum_{k=1}^4 v_k + (1-4\beta)v_0 \quad \dots(2-17b)$$

and the one dimensional Equation (2-8c) becomes:

$$\begin{aligned} \frac{v}{\Delta t} &= \beta \sum_{k=1}^2 v_k + (1-2\beta)v_0 \quad \dots(2-17c) \\ &= \beta (v_1 + v_2) + (1-2\beta)v_0 \end{aligned}$$

It has been demonstrated that certain stability limits impose conditions on the maximum values of β which can be used for computation (1) (2).*

e. Fluid and Electric Analogies. - It is of significant interest to compare the thermal principles and equations to the analogous set of equations governing the diffusion of water pressures in a confined fluid medium. This analogy gives rise to a simply realizeable physical model as described in Sections 2-01-f-(1) and 4-04-a below as well as in References 3 and 4.

Moreover, the same equations hold for the diffusion of currents and voltages in an electrical conducting medium which has a fixed capacitance to ground. This, too, may be realized in simple one and two-dimensional passive electrical circuits, as described in References 5, 6 and 10 in addition to the lumped equivalent as represented by the electronic analog computer outlined in Section 2-01-f-(2).

These analogies may best be summarized in tabular form as shown in Table I. The practical value of such analogies becomes obvious when the much greater manipulative flexibility of the fluid and electrical systems are considered. Moreover, instrumentation is significantly simpler and more precise in the analog media than in the prototype thermal system. However, one feature is distinctively absent, under normal conditions, in the fluid and electrical systems that indeed characterizes the present thermal problem and makes studies difficult: the presence of state changes

*Numbers in parentheses refer to Bibliography

TABLE I

THERMAL - FLUID - ELECTRIC ANALOGIES

ITEM	MEDIUM		
	THERMAL	FLUID	ELECTRIC
A-Variables (1)	Heat u	Volume S	Charge Q
(2)	Heat Flow q	Flow Q	Current i
(3)	Temperature v	Heat ^{al} H	Voltage e
B-Principles:			
Continuity (1)	$\frac{\partial u}{\partial t} + \nabla \cdot q = 0$	$\frac{\partial S}{\partial t} + \nabla \cdot Q = 0$	$\frac{\partial Q}{\partial t} + \nabla \cdot i = 0$
Conductivity (2)	$q = -k \nabla v$	$Q = -k \nabla H$	$i = -\sigma \nabla e$
Capacitance (3)	$\delta u = C dv$	$\delta S = A dH$	$\delta Q = C de$

and the concurrent latent heat effects. The analogy to this situation would require the equivalent capacitances to remain temporarily infinite until a corresponding amount of fluid volume or electrical charge had been transferred to or from the element. How this feature is secured in the hydraulic analog and in the electronic analog is described briefly in Sections 2-01-f-(1) and 2-01-f-(2), respectively.

f. Practical Methods of Computation. - Due to the complexity of the latent heat conditions in addition to the normally heterogeneous nature of the soil, and the randomness in the surface temperature distributions, it is not possible to obtain very many practically significant solutions to these problems in closed, mathematical form.

In order to overcome this handicap to rational analysis, several useful machine and hand computation methods are presented in the following section which are based on the principles outlined in the previous sections.

(1) Hydraulic Model. - By applying the analogy relationships of Section 2-01-e and the "lumping" approximations of Section 2-01-d, one finds that it is possible to solve a wide variety of thermal problems to an excellent approximation by means of a simple hydraulic model.

Basically, the hydraulic analog would consist of a series of small vertical chambers or wells connected to each other at the bottom through laminar tubes or orifices. Each chamber represents the heat content, both volumetric and latent, of a lump or layer of soil at a particular depth. Distance along the model corresponds to depth in the soil. The surface level of the liquid in the well is analogous to the temperature. To simulate the latent heat content of the layer an expansion chamber of suitable

area would be provided at a level representing the freezing temperature of the soil moisture. The size of this expansion, as well as the size of the vertical chambers, would be adjustable to correspond to the variations in latent heat and volumetric heat, respectively. More specifically, the sectional areas of the chambers are directly analogous to the C and L values, since storage volume S represents heat u and level H represents temperature v.

The interconnecting capillaries or laminar orifices are directly analogous to the conductive paths with the fluid conductance representing the thermal conductance. These conductances can be made adjustable in a simple fashion so as to readily permit wide variations in the values of k between layers.

The surface temperature variations are directly represented by variations in the level of an overflow-source tank at the end of the model which corresponds to the ground surface.

The flexibility of such a scheme is directly evident; moreover, the same model can be used to represent consolidation, seepage flow and other diffusion problems. Such a model had been proposed several years ago by Barron (4) and has been successfully used for latent heat problems by Bäckström (7). For diffusion problems associated with consolidation phenomena such a model has been effectively employed by one of the authors of this report (3).

(2) Electronic Analog Computer. - The hydraulic model just described becomes complex for 2-dimensional problems and 3-dimensional representations seem entirely impractical, due principally to the difficulties of interconnecting the chambers in a manner which will still permit direct observation.

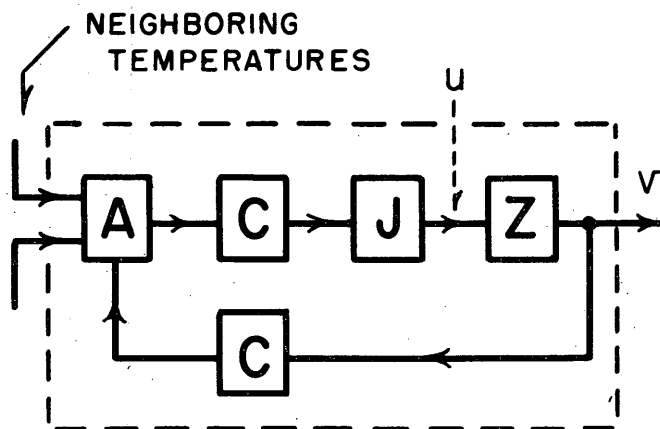
These disadvantages are not present in an electronic computer operating on essentially analogous principles. A 2-dimensional array of elements can be mounted on relay racks covering a relatively small amount of wall space.

The basic computing elements can be designed to solve the "lumped" approximate equations - the same as those which govern the behavior of the hydraulic model. For the sake of simplicity, only the circuits for a one-dimensional homogeneous computer will be outlined here, although, as stated, this method readily generalizes to the 2 and 3-dimensional cases.

In terms of operational blocks, the basic structure of the computer components representing one lump or layer of soil is shown in Figure 2A where the letters stand for the operations indicated beside the drawing. In particular, the "Z" - component represents the latent heat effect and has an input - output characteristic similar to Figure 1.

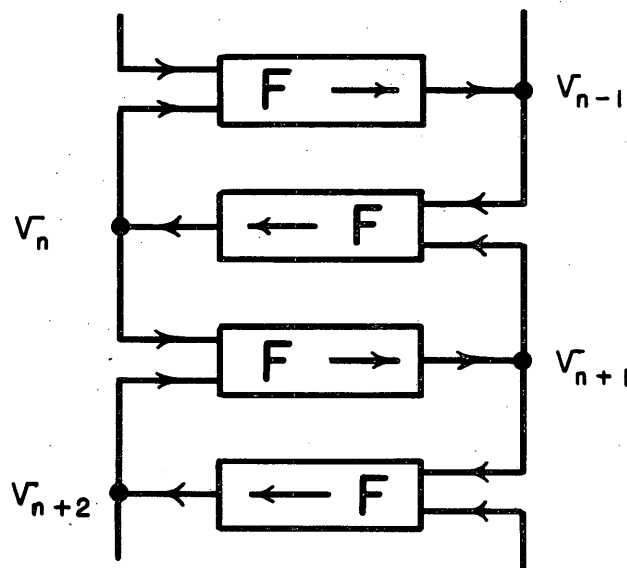
Thus each layer of soil is represented by an assemblage of computing components F which are interconnected as shown in Figure 2B to represent the entire soil mass.

This particular type of electronic computer is referred to as an active representation in which all variables are represented by analogous voltages in order to expedite the instrumentation and interconnection. Thus ground surface temperature fluctuations are represented by corresponding input voltage variations applied to the elements representing the top layer of soil.



SYMBOL	NAME	OPERATION
A	Adding Component	Output = $\sum(\text{Inputs})$
C	Coefficient Component	Output = $C \times (\text{Input})$
J	Integrating Component	Output = $\int (\text{Input}) dt$
Z	Inert Zone Component	(Similar to Figure 1)

A. ASSEMBLY "F" FOR ONE LAYER



B. INTERCONNECTION OF "F" COMPONENTS

SCHEMATIC BLOCK DIAGRAM ELECTRONIC ANALOG FROST COMPUTER

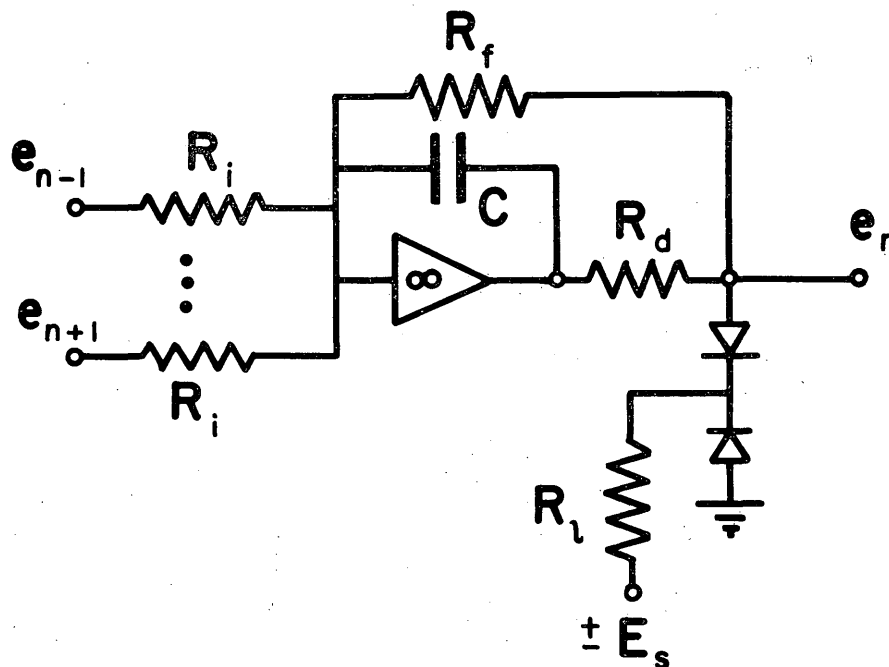
FIGURE 2

This particular representation is therefore a so-called "active" analog, in which all variables are voltages; in the absence of latent heat components, such computers are discussed in References 8 and 9. This type of analog computer is distinct from the more common "passive" analogs (5) (6), in which temperature (v) is represented by a voltage (e), but heat flow (q), by a current (i).

Such a computer can be made to operate on so-called "fast-time" to permit display on a cathode ray oscilloscope, or it may be run at "slow-time" (over a period of seconds or minutes) to permit recording on a single or multi-channel chart oscillograph. Actual surface temperature observations may be traced by a stylus input device which drives the computer, permitting direct check against field temperature and frost - thaw depth measurements.

The internal circuitry of a single "F" component is indicated in Figure 3. The input resistors (R_i) and the feedback resistor (R_f) correspond to the thermal resistance. The feedback capacitor (C) corresponds to the volumetric heat of the soil layer. The dropping resistor (R_d) determines the ratio of latent heat to volumetric heat. Since the computer may be arranged so that these circuit elements can be varied, it is possible to represent variations in soil thermal properties in a routine fashion.

The only changes in the above circuit required for 2 and 3-dimensional representation would be to provide four and six input resistors (R_i) respectively.



CIRCUIT DIAGRAM
SINGLE "F" COMPONENT
ELECTRONIC ANALOG FROST COMPUTER

FIGURE 3

(3) I.B.M. Solutions. -- As a means of obtaining reliable rational data on the freezing and thawing process as part of the second phase of the contract, and due to the non-availability of the hydraulic and electronic analog equipment just referred to, a number of computed studies were undertaken using the I.B.M. Card Programmed Calculator of the M.I.T. Statistical Services Division.

These studies had three main purposes, as follows:

- (1) Verification of the difference equations as outlined in Section 2-01-d, including the effects of latent heat.
- (2) Determination of the effect of the shape of the surface temperature-time curves on the total depth of freeze.
- (3) Investigation of the shapes of the depth-time curves during both the freezing and thawing period.

The basic numerical solution techniques have been extensively outlined in the available literature (11) (12) (13) (14) following in all essentials the procedures outlined in Section 2-01-d. However, to the best of the authors' knowledge no numerical solutions have been published involving the diffusion process when latent heat effects are significant.

In order to ascertain the precision of the numerical approximation procedure it seemed desirable to run check solutions to compare with the rational analysis of Section 2-02-c. Accordingly, twelve test solutions with various values of α and μ were carried out on the I.B.M. equipment. These are presented as Tables B-I through B-XII, inclusive, in Appendix B, and assume a step change in surface temperature.

Two I.B.M. solutions given as Tables B-XIII and B-XIV in Appendix B, display the effects of a sinusoidally varying surface temperature. They refer to purposes listed as items (2) and (3) above and are considered in Section 2-02-f.

These fourteen solutions, as well as several other check and trial solutions not presented, were obtained by automatic computation of the following set of equations and conditions:

$$S_{n, k+1} = S_{n, k} + \beta (e_{n-1, k} + e_{n+1, k} - 2e_{n, k}) \quad \dots (2-18)$$

$$\text{where } \left. \begin{aligned} e_{n, k} &= S_{n, k} - 1 && \text{for } S_{n, k} > 1 \\ e_{n, k} &= 0 && \text{for } 0 \leq S_{n, k} \leq 1 \\ e_{n, k} &= S_{n, k} && \text{for } S_{n, k} < 0 \end{aligned} \right\} \quad \dots (2-19)$$

The set of conditions (2-19) defines a normalized version of the internal energy - temperature relation similar to Figure 1, with e being a measure of the temperature v and S a measure of the heat content u . Rendering the relation in this simple form permitted a satisfactory programming of the calculation. However, it makes necessary the proper scaling of variables and interpretation of results, as indicated below and in Appendix B.

The dimensionless parameter β is defined in Section 2-01-d

as

$$\beta = \frac{a \Delta t}{\Delta s^2}$$

where a is the (thermal) diffusivity, Δt , the time increment, and Δs , the depth increment. For stable calculations in the one-dimensional case β must be taken equal to or less than 0.5. The values chosen for the various I.B.M. solutions are indicated in Appendix B.

The relation between the computed e - values and the actual temperatures $v(t)$ is given by the expression:

$$v(t) = v_s \cdot \frac{e_k}{\mu} = \frac{F}{T} \frac{e_k}{\mu} \quad \dots(2 - 20)$$

where $v_s = F/T$ is the actual or equivalent step change in surface temperature below freezing and $\mu = Cv_s/L$. In other words, the e - values bear the same relation to the parameter μ that the v - values bear to v_s . The time t corresponding to any k - value is given by the defining relation:

$$t = k\Delta t \quad \dots(2 - 21)$$

while the depth x corresponding to any n - value is given by the definition:

$$x = n\Delta s \quad \dots(2 - 22)$$

Thus instantaneous temperature gradients as well as temperature-time curves can be plotted from the values given in Tables B-I to V-XIV just as for actual thermocouple data. Such curves were drawn by the authors and the conclusions are given in Section 2-02-f.

(4) Hand Computations. - Using procedures very similar to those outlined in the previous section, it is possible to carry out numerical solutions using a slide rule or desk calculator. In fact, for certain cases completely graphical procedures are practical.

Efficient tabular forms for such numerical solutions are detailed in the available literature (11) (12) (13) (14) while the Binder graphical procedure for heat flow problems is outlined by Jakob (15).

However, the procedures of the cited references must be modified to account for the latent heat effects. This involves use of the $e - s$ relationship outlined in the previous section. A portion of a sample calculation sheet is given in Table II. The calculations follow directly from Equations (2 - 18) and (2 - 19) as indicated.

TABLE II
SAMPLE HAND COMPUTATION OF FROST PENETRATION
(A) NUMERICAL CALCULATIONS:

$h \backslash k$	0	1	2	3
0	-1.40	-1.48	-1.50	-1.48
	-1.06 - .42	- .90 - .36	- .79 - .32	- .73 - .29
1	- .34 - .34	- .58 - .58	- .71 - .71	- .75 - .75
	- .46 - .18	- .58 - .23	- .71 - .28	- .75 - .30
2	+ .12 +1.12	0 + .97	0 + .81	0 + .58
	- .08 - .03	- .18 - .07	- .12 - .05	- .09 - .04
3	+ .20 +1.20	+ .18 +1.18	+ .12 +1.12	+ .09 +1.09
	- .02 - .01	- .02 - .01	- .06 - .02	- .05 - .02
4	+ .22	+ .20	+ .18	+ .14

(B) KEY TO CALCULATIONS:

	k	$k+1$	
$n-1$	$e_{n-1,k}$	-	-
	$e_{n-1}-e_{n,j} \beta(e_{n-1}-e_n)$	-	-
n	$e_{n,k}$ $S_{n,k}$	$e_{n,k+1}$ $S_{n,k+1}$	-
	$e_n-e_{n+1,j} \beta(e_n-e_{n+1})$	-	-
$n+1$	$e_{n+1,k}$	-	-

2-02. DEPTH OF FROST PENETRATION.

a. Synopsis. - This article outlines the writers' development of a rational formula to predict the depth of frost penetration which takes into consideration nearly all of the factors of primary significance to the problem. The nomograph included in this investigation employs this formula as its basis.

This formula is premised on obtaining the exact solution to an idealized problem which is sufficiently representative of actual physical conditions to serve as a basic reference for any future correction terms.

Following the derivation of the rational formula, the writers' present the results of their statistical studies of actual frost penetration data. It is concluded that the rational formula generally yields better results than other expressions which were studied.

The latter parts of this section are devoted to a study of the effect of a cyclic surface temperature fluctuation on the depth of frost penetration and the rate of freezing and thawing. Finally, the multilayered system is treated.

b. Review of Literature. - As a first step in these studies, existing literature pertaining to prediction of the depth of frost penetration was reviewed. The principle objective was to determine the most suitable formula on which to base the nomograph. Consideration was given to three criteria:

- (1) The mathematical validity of the existing formulas;
- (2) The ease of application of each formula;

(3) The statistical reliability in predicting depths of frost penetration as compared to field observations.

In particular, four formulas were considered, namely:

$$X = \sqrt{\frac{48kF}{L}} \quad \dots(2 - 23)$$

$$X = \sqrt{\frac{48kF}{L + \gamma_d v_o (C_s + 0.01w)}} \quad \dots(2 - 24)$$

$$X = \sqrt{\frac{48kF}{L + C \left(v_o + \frac{F}{2t} \right)}} \quad \dots(2 - 25)$$

$$X = \sqrt{\frac{24kF}{L + C \left(v_o + \frac{F}{2t} \right)}} \quad \dots(2 - 26)$$

The first equation is the Stefan equation which considers latent heat only. The second equation is the so-called "Minneapolis" formula. The third and fourth expressions are formulas which have been studied by the Frost Effects Laboratory of the New England Division of the Corps of Engineers. These last three consider certain of the significant volumetric heat effects by making various rough approximations.

After a preliminary analytical and statistical review of the above four formulas, the writers concluded that Equation (2 - 25) was the most sound. Other statistical studies⁽¹⁶⁾ comparing actual depths of frost penetration against those computed from the above equations had indicated that Equation (2 - 25) gave reasonably reliable results.

However, the writers were not entirely satisfied with this equation, since it did not account for the volumetric heat effect in the unfrozen ground, which causes a substantial reduction in frost depths in the more

temperate climates. Accordingly, they undertook the derivation of a more rational formula as outlined in Section 2-02-c and Appendix A. This was based on the earlier work of Neumann, as reported in Carslaw and Jaeger⁽¹⁷⁾, dealing with melting and solidification problems in still water.

After this formula was developed, it was discovered to be identical in nature to a formula developed by W. P. Berggren⁽¹⁸⁾, which has been expressed in the form:

$$X = 2B \sqrt{at} \quad \dots(2 - 27)$$

by Shannon⁽¹⁹⁾. The relation between the factor B and the notation of the writers is outlined at the end of Appendix A. The unfavorable report by Shannon in the reference cited had caused the writers to be initially mislead; the 50% discrepancies mentioned in connection with Shannon's calculations must be attributed largely to inadequate data on the thermal properties of the soils considered.

The statistical studies, undertaken by the writers and by the Frost Effects Laboratory as reported in Section 2-02-d-(2), testify strongly to the general reliability of this modified Berggren formula.

c. Derivation and Interpretation of Rational Formula. - A detailed derivation for the writers' rational formula for the depth of frost penetration is given in Appendix A. The most useful form for using the formula is given below as Equation (2 - 29).

As indicated in Appendix A, it is assumed that the soil is a semi-infinite mass of uniform properties and having initially a uniform temperature. It is further assumed that the surface temperature is suddenly changed from its initial value v_0 above freezing to a temperature v_s below freezing.

The equations which must hold for this problem are the diffusion equations (2 - 8c) in both the frozen and unfrozen soil as indicated in Figure 4. The latent heat property becomes a continuity condition which states that the rate of flow of heat in the frozen soil at the frost interface must be equal to the sum of the rate of heat flow in the unfrozen soil at the interface and the heat given off at the interface when the soil moisture freezes.

The solution of these equations may be written in the form:

$$X = \lambda \sqrt{\frac{48kv_s t}{L}} \quad \dots(2 - 28)$$

For practical computations for the depth of frost it is generally assumed that v_s represents the average surface temperature below freezing during the freezing period. Thus, if F represents the air freezing index and nF the surface freezing index, then:

$$nF = v_s t$$

and therefore:

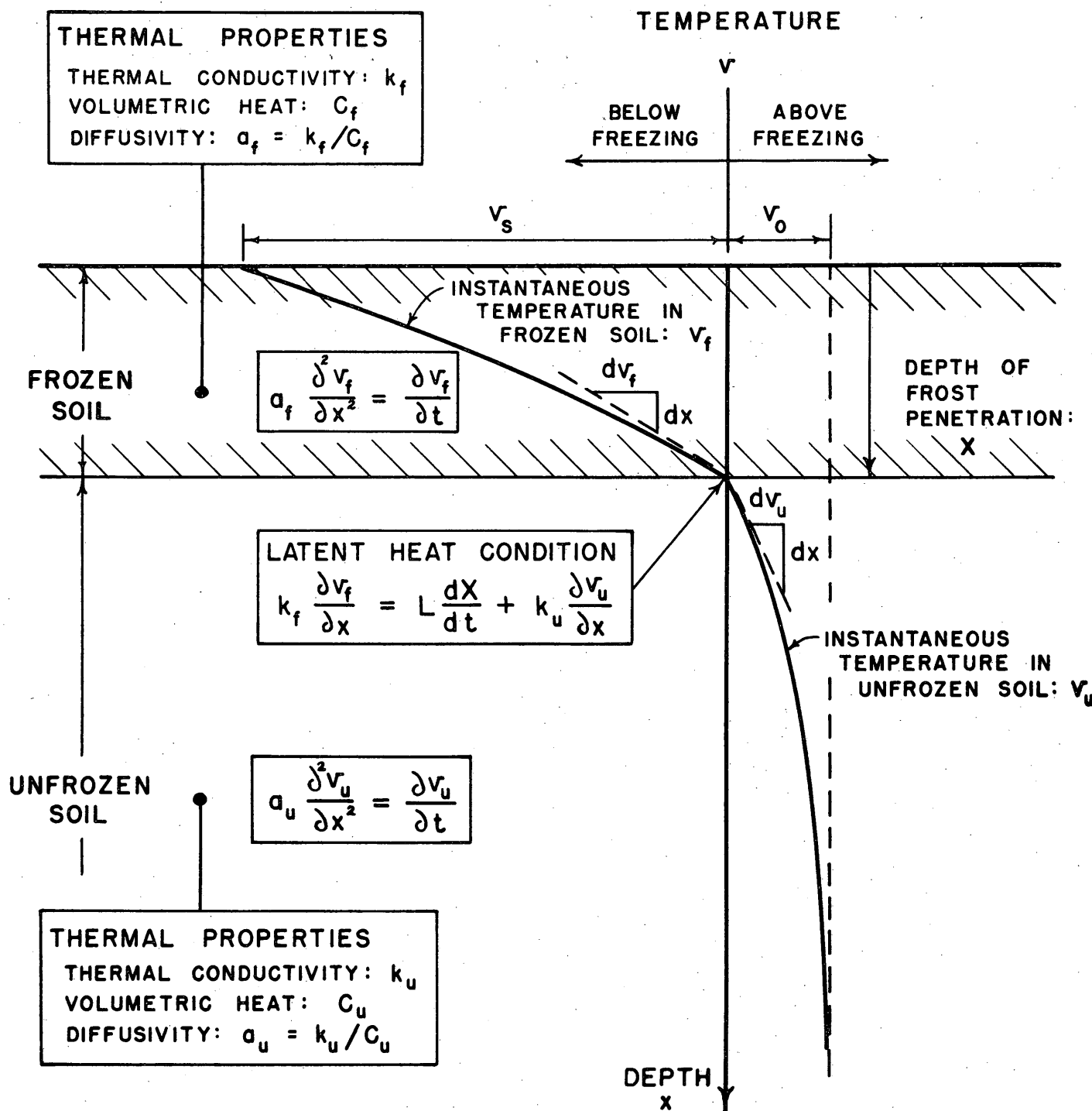
$$X = \lambda \sqrt{\frac{48knF}{L}} \quad \dots(2 - 29)$$

which will be called the modified Berggren formula.

The term λ , a correction coefficient modifying the square-root term, which is merely the familiar Stefan equation, is a complicated function of three dimensionless parameters α , μ and δ . It has been shown that for all practical purposes δ may be assumed equal to 1.0 in which case λ is given graphically by the curves in Figure 5 as a function of α and μ where:

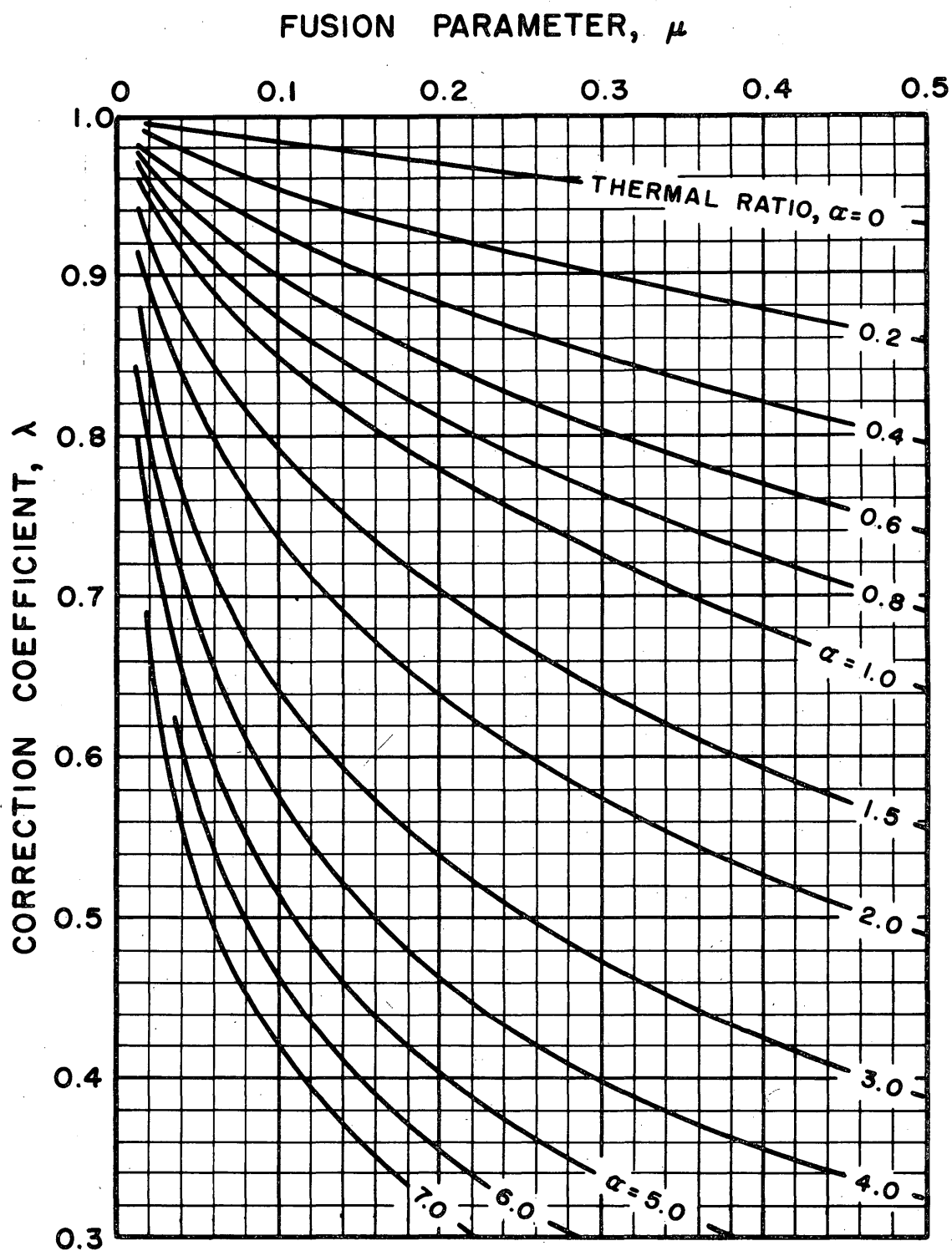
$$\text{thermal ratio, } \alpha = \frac{v_o t}{nF} = \frac{v_o}{nv_s} \quad \dots(2 - 30)$$

$$\text{and fusion parameter, } \mu = \frac{CnF}{Lt} \quad \dots(2 - 31)$$



THERMAL CONDITIONS DURING FROST PENETRATION

FIGURE 4



**CORRECTION COEFFICIENT
IN THE
MODIFIED BERGGREN FORMULA**

FIGURE 5

The thermal ratio α is a measure of the ratio of the initial ground temperature, or in practice, the mean annual temperature above the freezing point, to the time-average surface temperature below freezing during the freezing period. It is apparent, then, that α decreases in the colder climates. In parts of Alaska α may be zero or actually negative.

The fusion parameter μ is a measure of the heat removed in the frozen soil below the freezing point as compared to the latent heat of the soil moisture. As the latent heat term becomes very large, μ approaches zero. It is seen, furthermore, that μ varies with the freezing index and is therefore largest in the arctic climates.

d. Comparison of Rational Formula with Other Formulas. - A comparison of Equation (2 - 29) with other formulas proposed and used to compute the depth of frost penetration will be made at two levels. First, the assumptions made in each derivation are outlined and a general evaluation of the effect of these assumptions on computed results is presented. Second, results of statistical studies using actual frost penetration data are analyzed.

(1) General. - Equations (2 - 23) (2 - 25), and (2 - 26) of Section 2-02-b will be used for this comparison. It can be shown that all three expressions can be rearranged and written in terms of a correction coefficient which is some function of α and μ as defined in the previous section. Thus:

$$X = \lambda \sqrt{\frac{48 \text{ kmF}}{L}}$$

where λ is given by the following expression for formulas (2 - 23), (2 - 25) and (2 - 26) respectively:

$$\lambda_1 = 1.0 \quad \dots(2 - 32)$$

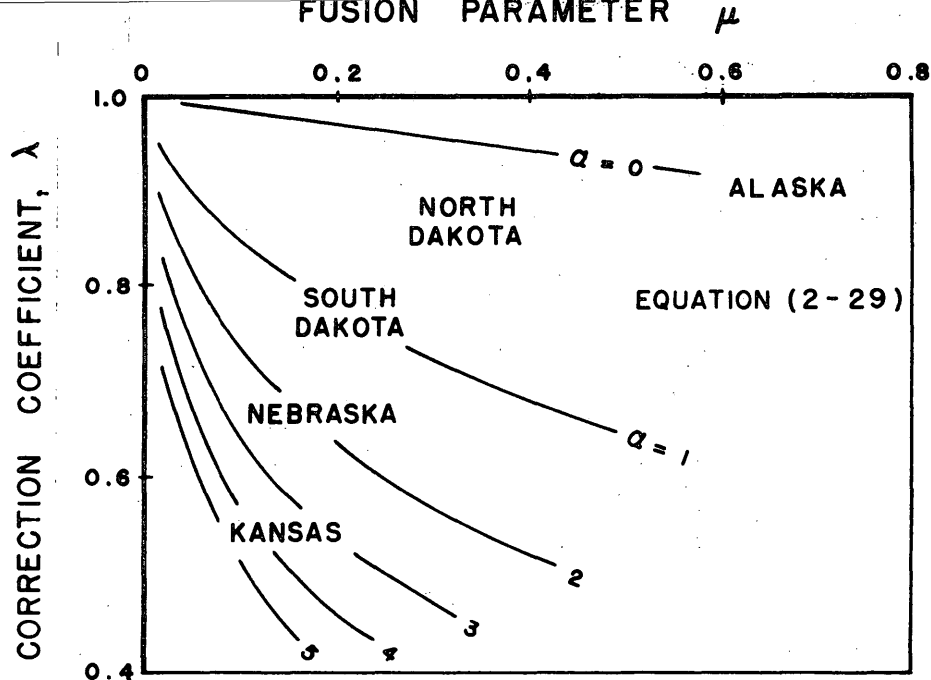
$$\lambda_2 = \frac{1}{\sqrt{1 + \mu(\alpha + 0.5)}} \quad \dots(2 - 33)$$

$$\lambda_3 = \frac{0.707}{\sqrt{1 + \mu(\alpha + 0.5)}} \quad \dots(2 - 34)$$

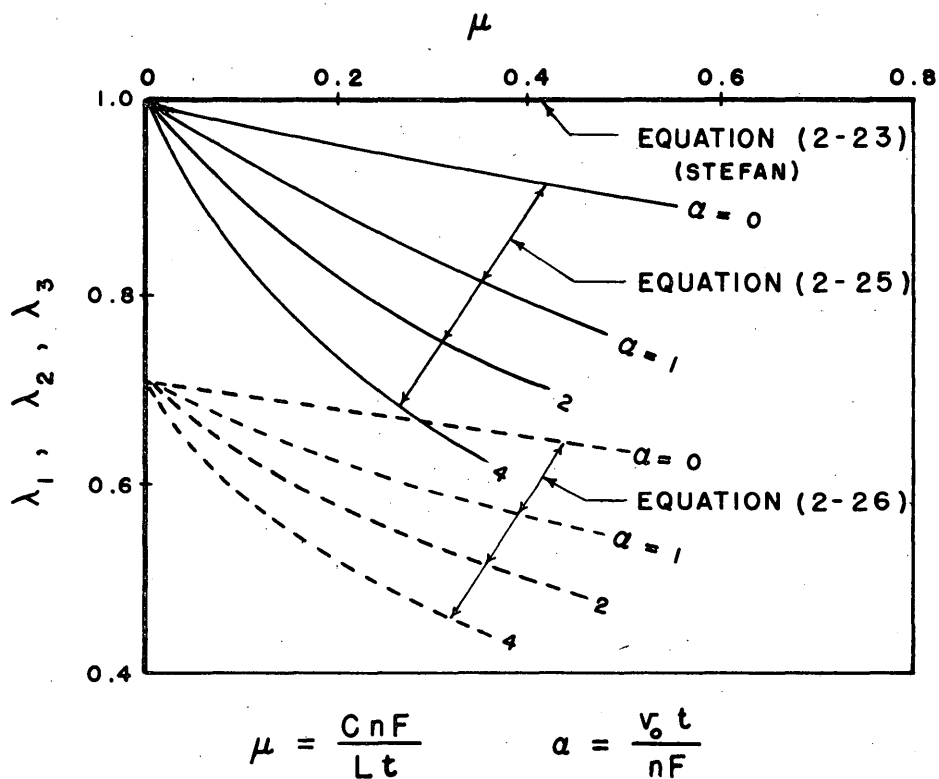
Curves for λ_1 , λ_2 and λ_3 are shown in Figure 6B.

Since the Stefan equation (2 - 23) ignores the volumetric heat given off as the soil temperature is lowered to and below the freezing point, μ is equal to zero and therefore $\lambda = 1.0$ for all α . Computed depths of frost penetration based on this formula will generally be too great. This is especially true in temperate regions where actual values of λ may be 0.6 or smaller. From Figure 6A it is seen that in temperate Kansas, for example, the correction coefficient is perhaps 0.55 which means that the actual depth of frost penetration would be about 55% of the value determined from the Stefan equation. On the other hand, in Alaska the equation would be expected to yield good results.

Equations (2 - 25) and (2 - 26) consider the effects of volumetric heat under various assumptions. They should be expected to yield somewhat better predictions than the Stefan equation. It can be seen by comparing Figures 6A and 6B that for small α equation (2 - 25) gives good results while equation (2 - 26) with a much smaller λ will predict too shallow. An engineer in Kansas or even Nebraska where α is large would find that equation (2 - 26) was very good while equation (2 - 25) predicted too deep.



A. λ, α, μ CURVES FOR EQUATION (2-29)



B. λ, α, μ CURVES FOR OTHER EQUATIONS

COMPARISON OF λ, α, μ CURVES

FIGURE 6

The statistical studies described in the following section generally confirm the above statements.

(2) Statistical Studies. - The following studies are aimed at comparing predicted depths of frost penetration given by the modified Berggren formula (2 - 29) with actual depths determined by thermocouple readings and test pits. These results are compared with predicted results, using Equations (2 - 25) and (2 - 26), the latter including the pavement thickness. Data for this study have been taken from 48 cases given in reference (16).

Two studies were conducted by the writers. Average values of C , L and k as given in reference (16) have been used. In addition, the air freezing index has been used for both studies.

The first analysis, involving the modified Berggren formula only, is based on per cent variation between observed and predicted depths. The following results, then, may be compared directly with the results of studies conducted at the Frost Effects Laboratory and reported on page 13 of reference (16).

Mean	+15.0%
Median	+ 7.3%
Mode	+ 2.0%
Standard Deviation	+28.8%
Skewness	+ 0.451
Sum of Squares	5535

No. of predicted values within ± 6 in. of observed depth

The graphical representation of the dispersion is shown in Figure 7. This may be compared with plate 5 of reference (16).

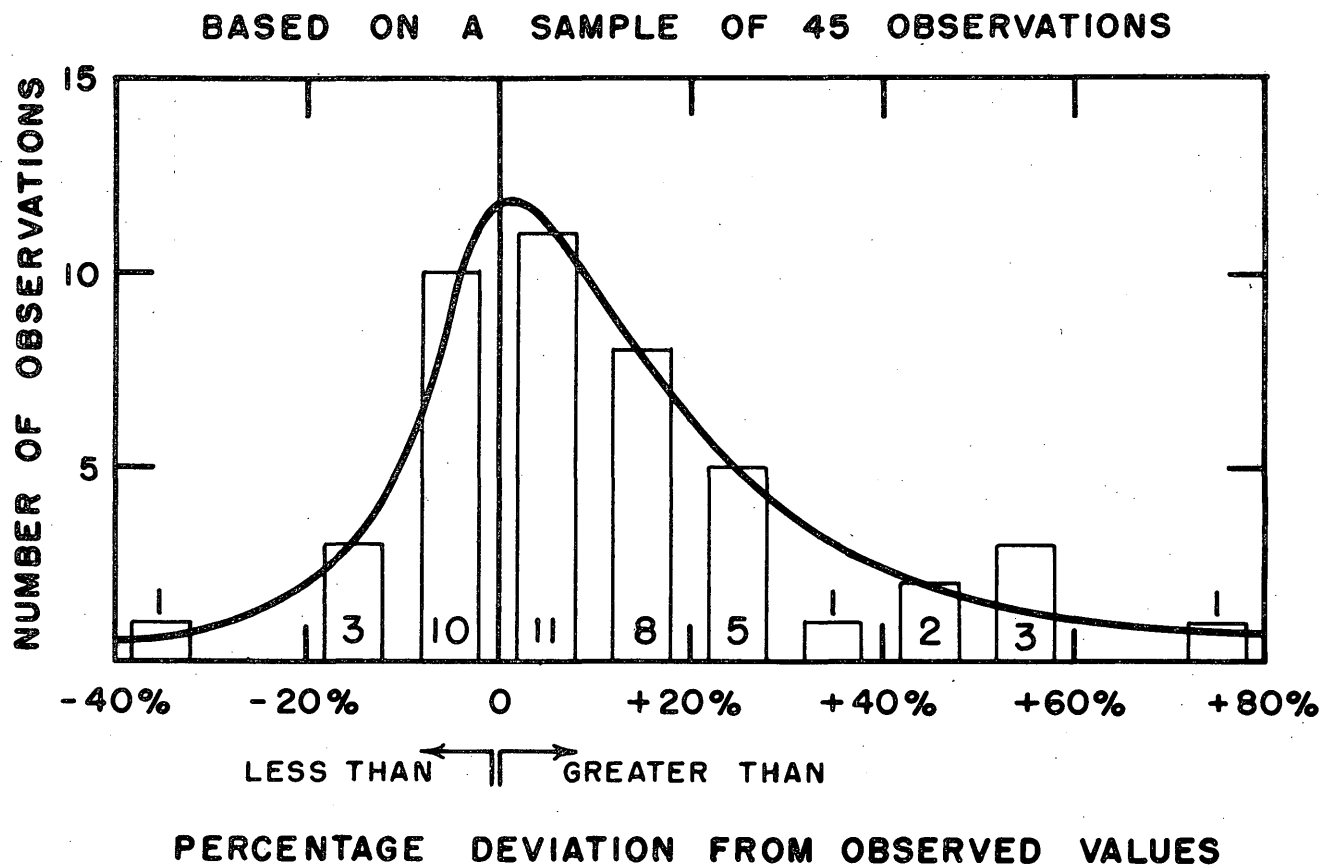
In general these results do not show a prediction which is significantly superior to the other equations.

The second analysis is based on inches variation between observed and predicted depths, comparing Equation (2 - 25), Equation (2 - 26) (with pavement thickness added) and Equation (2 - 29). For this study the data have been arranged in two groups, one with the 24 highest values of the actual correction coefficient λ_a^* (.915 to .715) and the other with the lowest 24 (.714 to .295). The following results were obtained:

	Group I		Group II	
	λ_a (.915 to .715)		λ_a (.714 to .295)	
	Avg. Obs. Depth = 52.3 inches		Avg. Obs. Depth = 40.9 inches	
	Avg. Deviation from Observed Depth	Standard Dev. from the Avg.	Avg. Dev. from Obsr'd. Depth	Std. Dev. frm. the Avg.
Eq. (2 - 25)	+5.4"	+4.3"	+19.2"	+9.8"
Eq. (2 - 26)	-4.4"	+5.9"	+ 7.1"	+8.4"
Eq. (2 - 29)	-0.4"	+4.5"	+12.2"	+8.1"

It is apparent from the above table that in general for high λ values Equation (2 - 25) predicts frost depths which are too great while Equation (2 - 26) predicts too shallow. For low λ , all equations predict too deep. Equation (2 - 25) predicts about 50% too deep while Equation (2 - 26) is less than 20% in error. These results, then, confirm the general comparisons made in the previous section.

$$* \lambda_a = \frac{\text{observed depth of freezing}}{\sqrt{\frac{48kF}{L}}}$$



DISPERSION OF COMPUTED FROST DEPTHS USING MODIFIED BERGGREN FORMULA

FIGURE 7

To be more specific, one may readily see from the above table that for the first 24 points (Group I) the modified Berggren formula (2 - 28) predicts the depth of penetration within an average deviation of -0.4 inches out of a group average observed depth of 52 inches, with a probable error of +3.0 inches (0.6745×4.5). Thus the average deviation for the data of Group I is less than 1% of the average depth and is better than 10 times smaller for the new formula than for Equations (2 - 25) or (2 - 26). This means that, on the average, for values of λ between 1.0 and 0.707 (say), one may expect Equation (2 - 29) to predict about 82 per cent of the observations within 6 inches of the actual values. For example, in Group I of the data studied, 21 out of 24 points or 87 per cent were within 6 inches.

For the second 24 points (Group II), Equation (2 - 29) predicts the depth of penetration within an average deviation of +12.2 inches out of a group average observed depth of 41 inches with a probable error of +5.5 inches. Then, on the average, for values of λ less than 0.707, one may expect the new formula to predict only about 22 per cent of the observations within 6 inches of the observed values. For example, in Group II, 5 out of 24 points or 21 per cent were within 6 inches.

Since, in Group II, for all the formulas, both the average deviations and the probable errors are significantly large, one is led to suspect that at least part of the fault lies with the assumed data. From a superficial analysis of this group, it would appear that the actual values of the thermal conductivity k are in some cases significantly different from the value assumed for this and the previous studies reported in Reference (16). Moreover, the assumption of $k_u = k_f$ seems to be appreciably

in error for certain of these points. These conclusions would suggest that for those cases in which the approximate values of α and μ point to values of λ less than 0.707, a more refined determination of reliable soil constants is in order.

Following a suggestion by the writers, the Arctic Construction and Frost Effects Laboratory made new statistical studies using for each case an average thermal conductivity k determined from the water content and dry density of the soil. Results of these studies showed that Equation (2 - 29) gave predicted results superior to other formulas. The Arctic Construction and Frost Effects Laboratory therefore approved the writers' recommendation that the nomograph called for in the first phase of this research be based on the rational formula (2 - 29) developed by the writers.

e. Limitations and Applicability of Formulas. - Before discussing the limitations of Equation (2 - 29) it is of value to list again the basic assumptions made in the derivation of the formula:

- (1) The soil mass was considered to be homogeneous and one-dimensional.
- (2) The initial temperature in the soil mass was assumed to be everywhere constant at the value v_0 above freezing.
- (3) The surface temperature was assumed to change suddenly from v_0 above freezing to v_s below freezing, and remain steady at this latter value.
- (4) The effect of latent heat was considered to introduce a heat source (or sink) at the moving freezing point interface, with a heat flow proportional to the speed of motion.

(5) Under the above conditions an exact solution was obtained, taking into account both volumetric heat and latent heat, as well as the variations in thermal properties upon freezing of the soil moisture.

This exact formulation of the above idealized case was expressed in terms of three parameters, namely the thermal ratio, α , the fusion parameter, μ , and the root diffusivity ratio, δ . These same constants would govern the behavior in soil freezing problems under the much more complex conditions encountered in actual practice. Thus it may be expected that the depth of frost penetration in these actual cases is affected in much the same way by α , μ , and δ , as the idealized case for which the formula was derived.

However, the failure to realize these simplifications may produce significant divergences; therefore, the most important of these deviations are outlined in the following paragraphs.

The actual surface temperature variations are not uniform over the surface area, but rather are appreciably affected by ground cover, pavement type, etc., as well as by local variations in wind velocity, shielding, snow cover, etc. For any measure of success using one-dimensional methods of prediction, considerable care must be taken to derive a set of consistent and truly representative conditions and equal care must be taken in the interpretation of formula results for actual cases.

All actual soils are layered or otherwise variegated to a greater or lesser extent and the assumption of homogeneity must be accepted as a mere approximation. As the variations from layer to layer become more pronounced the formula predictions may become seriously in error unless

deliberate care is taken to compensate for the layering as outlined in Sections 2-02-g and 3-02-c.

The initial temperature in the soil mass upon the start of freezing is not uniform at a value v_0 above freezing, but rather varies in some non-uniform way from the surface temperature down to a temperature comparable to the mean annual temperature at a point sufficiently deep in the soil. Representative of such variations are the values at the start of freezing given in Tables B-XIII and B-XIV. However, as testified by the I.B.M. solutions themselves, as well as field data and the analyses of Section 2-02-f, the effects of such non-uniformity of temperature on the correction factor, λ , are not appreciable so long as v_0 is taken as the mean annual temperature.

Much of the same reasoning applies for the shape of the surface temperature-time curve and its effects on penetration depths, as discussed in Section 2-02-f. Again, as long as the equivalent step change v_g is taken as the mean temperature below freezing during the freezing period, a good agreement exists between actual seasonal depths and those predicted by the formula.

Other limitations and modifications of the formula as applied to practical problems of prediction are discussed in connection with the nomograph in Sections 3-02 and 3-03.

f. Depth -Time Curves. - While Equation (2 - 29) gives consistent predictions of seasonal depths of frost penetration, attempts to calculate the depth-time curves based on partial freezing indices are likely to encounter substantial errors. This may be demonstrated roughly as in the following paragraphs.

If the soil is initially at a uniform temperature v_0 and a surface temperature variation $v_s(t)$ below freezing is applied, then to a crude first approximation the following equation may be shown to hold:

$$\frac{x^2}{2} = \frac{k}{L} \int \frac{v_s(t) \cdot dt}{1 + \frac{c_u v_0}{L} + \frac{1}{2} \frac{c_f v_s(t)}{L}} \quad \dots(2 - 35)$$

This equation accounts for the latent heat effect, and the most significant terms due to volumetric heat effects. It may be normalized to the form:

$$\phi^2(t) = \frac{x^2(t)}{x_m^2} = \frac{\int_0^{\tau} \left(\frac{\epsilon}{1+\rho\epsilon} \right) d\tau}{\int_0^1 \left(\frac{\epsilon}{1+\rho\epsilon} \right) d\tau} \quad \dots(2 - 36)$$

$$x_m = \lambda \sqrt{48 \frac{KF}{L}} \quad \dots(2 - 37)$$

$$\lambda^2 = \frac{1}{1+\sigma} \int_0^1 \left(\frac{\epsilon}{1+\rho\epsilon} \right) d\tau \quad \dots(2 - 38)$$

where $\sigma = \alpha\mu = c_u v_0 / L$

$$\rho = \frac{1}{2} \mu / 1 + \alpha\mu$$

$$\epsilon(\tau) = v_s / (F/T)$$

T = Duration of Freezing Index (in DAYS)

$$F = \int_0^T v_s(t) dt = \text{Freezing Index (in DEGREE-DAYS)}$$

$\tau = t/T$ = Relative Time during Freezing Period

(i.e. $\tau = 0$ corresponds to start of freeze

$\tau = 1$ corresponds to end of freeze.)

Then $\phi(\tau) = X(\tau)/x_m$ measures the relative SHAPE of the depth-time curve.

From the definitions above it is necessary that:

$$\int_0^1 \epsilon d\tau = 0$$

whence it becomes evident that the function $E(\tau)$, also, merely measures the SHAPE of the surface temperature curve during the freezing period. Thus we may investigate simply using Equations (2 - 36) and (2 - 38) the effects on the depth-time curves of various surface temperature time curves.

The simplest such case is that of the constant step changes. In this case $E = 1.0 =$ constant throughout the period and the integral for λ becomes

$$\lambda^2 = \frac{1}{1+\sigma} \int_0^1 \frac{d\tau}{1+\rho} = \frac{1}{(1+\sigma)(1+\rho)} \quad \dots(2 - 39)$$

Upon substituting the defining constants for the parameters σ and ρ , this expression for λ is seen to give the value corresponding to Equation (2 - 25), namely:

$$\lambda = \frac{1}{\sqrt{1+\alpha(\mu+0.5)}} \quad \dots(2 - 40)$$

Thus the problem of investigating the effects of the surface temperature curve on the frost penetration depth curve may be factored into two parts, namely:

- (a) The SHAPE effect (ϕ) Equation(2 - 36)
- (b) The MAGNITUDE effect (λ) Equation(2 - 38)

It is easy to show that for any shape $E(\tau)$ symmetric about the midwinter point $\tau = 0.5$, which is very nearly true for all practical instances, then λ may be very closely related to the mean value $\bar{E} = 1.0$ by the following expression:

$$\lambda^2 \approx \frac{1}{1+\sigma} \frac{\bar{E}}{1+\rho\bar{E}} = \frac{1}{(1+\sigma)(1+\rho)} \quad \dots(2 - 41)$$

This fact would indicate that the shape of the surface temperature curve during the freezing season has only a very slight effect on λ , and therefore on the total depth of freeze. This hypothesis has been substantiated both from field measurements and by the I.B.M. studies referred to in Section 2-01-f-(3).

On the other hand, a slight effect on the shape of the depth-time curve does exist and may be explored by substituting the above expression for λ into the equation for ϕ to give the form:

$$\phi(z) = \sqrt{(1+\rho) \int_0^z \left(\frac{\epsilon}{1+\rho\epsilon} \right) dz} \quad \dots(2-42)$$

The integral may be approximated by a finite sum in the form:

$$\phi_K = \sqrt{(1+\rho)h \sum_K \frac{\epsilon_K}{1+\rho\epsilon_K}} \quad \dots(2-43)$$

where $h = \Delta z$ and where $\int_0^1 \epsilon dz = 1.0$ is replaced by the condition $h \sum_K \epsilon_K = 1.0$. Practical values of ρ are of magnitudes not much greater than 0.2. Accordingly, the table below has been calculated with $h = 0.1$ and values of ϵ_K and ρ as indicated, in order to show the effect of ρ on ϕ .

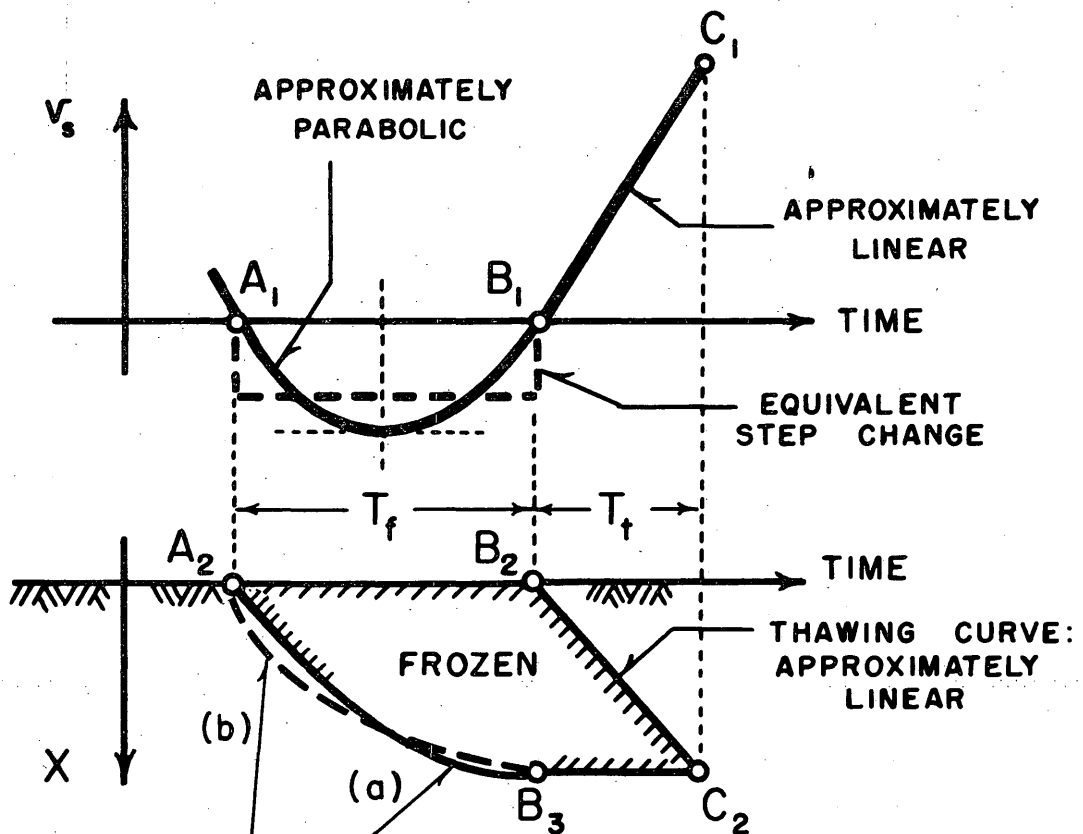
From Table III, it is clear that the parameter ρ has no significant effect upon the shape of the depth-time curve over its physically practical range. This permits one to draw valid conclusions as to the shape of the depth-time curves from the simple case where only latent heat is important ($\rho = 0$). Since ϕ^2 is then merely proportional to the freezing index, such cases may be explored very quickly. Conclusions based upon this approximation are shown in Figure 8.

TABLE III
SHAPE OF DEPTH-TIME CURVES

VALUES OF ϕ_k

BASED UPON A PARABOLIC DISTRIBUTION OF e_k

k	e_k	VALUES OF ρ			
		0	0.1	0.2	0.4
1	0.29	0.17	0.17	0.18	0.19
2	0.76	0.32	0.32	0.33	0.35
3	1.12	0.47	0.47	0.48	0.48
4	1.36	0.59	0.60	0.60	0.60
5	1.47	0.71	0.71	0.71	0.70
6	1.47	0.80	0.81	0.80	0.80
7	1.36	0.88	0.89	0.88	0.88
8	1.12	0.95	0.94	0.94	0.94
9	0.76	0.99	0.99	0.98	0.98
10	0.29	1.00	1.00	1.00	1.00



FREEZING CURVES :

- (a) FOR PARABOLIC TEMPERATURE:
APPROXIMATELY PARABOLIC WITH
VERTICAL AXIS THROUGH B_3
- (b) FOR STEP TEMPERATURE:
APPROXIMATELY PARABOLIC WITH
HORIZONTAL AXIS THROUGH A_2

SHAPE OF DEPTH-TIME CURVES

FIGURE 8

g. Multilayered Systems. -- In any given layer, m, the instantaneous heat flow, assuming a linear gradient, is given by Equation (2 - 10) or:

$$q_m = -k_m \frac{\Delta v_m}{d_m} \quad \dots(2 - 44)$$

which may be rearranged to solve for the temperature difference in terms of flow and the thermal resistance of the layer, $R_m = d_m/k_m$, to give:

$$\Delta v_m = R_m q_m \quad \dots(2 - 45)$$

For an entire series of layers, from the surface to layer n, the total temperature differential may be given by:

$$v_s - v_n = \left(\sum_{m=1}^n R_m \right) q_n \quad \dots(2 - 46)$$

If the n - th layer is at the freezing temperature, then $v_n = 0$.

Moreover, if the only source of heat is considered to be the cooling of the layer from v_o to the freezing point and subsequent revival of the latent heat L_n , then the total heat removed to freeze the layer n is given by:

$$U_n = [L_n + C_n v_o] d_n \quad \dots(2 - 47)$$

Thus the equation for the freezing of this layer can be written:

$$q_n = \frac{dU_n}{dt} \quad \dots(2 - 48)$$

or

$$dU_n = q_n dt = \frac{v_s}{R_t} \cdot dt \quad \dots(2 - 49)$$

In terms of the partial freezing index $F_n = \int_{t_{n-1}}^{t_n} v_s dt$ and realizing that $R_t = (\sum_{n=1}^{n-1} R) + \frac{1}{2} R_n$, the expression may finally be written as:

$$F_n = U_n \left[R_1 + R_2 + \dots + \frac{R_n}{2} \right] \dots (2 - 50)$$

For a double-layered system, then, the total freezing index will be:

$$F = F_1 + F_2 = R_1 \left(\frac{U_1}{2} + U_2 \right) + R_2 \frac{U_2}{2} \dots (2 - 51)$$

$$\text{since } F_1 = U_1 \frac{R_1}{2}$$

$$F_2 = U_2 \left(R_1 + \frac{R_2}{2} \right)$$

For a triple-layered system this gives:

$$F = R_1 \left(\frac{1}{2} U_1 + U_2 + U_3 \right) + R_2 \left(\frac{1}{2} U_2 + U_3 \right) + R_3 \left(\frac{1}{2} U_3 \right) \dots (2 - 52)$$

For the general case this becomes:

$$F = \sum_{k=1}^n R_k \left[\frac{1}{2} U_k + \sum_{m=k+1}^n U_m \right] \dots (2 - 53)$$

This may also be written:

$$F = \sum_{k=1}^n U_k \left[\frac{1}{2} R_k + \sum_{m=1}^{k-1} R_m \right] \dots (2 - 54)$$

These last equations are employed in Section 3-02-c in connection with the applicability of the nomograph to multilayered systems. In this instance $U_n = L_n d_n$ accounting for latent heat effects only, since the volumetric heat effects are contained in the effective λ value.

PART III. CONSTRUCTION AND USE OF NOMOGRAPH

3-01. DESCRIPTION OF THE NOMOGRAPH. -

A copy of the nomograph as prepared by the authors as part of this research is included in an envelope on the inside of the back cover. This section of the report will serve to explain its physical makeup, to cite directions for its use, and to discuss its limitations and assumptions.

The nomograph consists principally of three figures. In addition, a table of notation, a list of equations, and an example of its use are included on the printed sheet.

Figure 1: Figure 1 is a composite of nomographs already published (20, plate 16) for the determination of the volumetric heat of the unfrozen and frozen soil C_u and C_f , and for the evaluation of the latent heat of fusion L . One modification has been made, however, that of assuming a value of 0.17 BTU/lb for the specific heat of dry solids (instead of 0.20) in the equation for C . The value of 0.17 appears to represent a more realistic figure for soil temperatures near the freezing point (21, page 71).

Figure 2: Figure 2 presents curves for determining the thermal conductivity of the frozen soil, k_f , and unfrozen soil, k_u . These curves are the results of investigations made at the University of Minnesota under the immediate supervision of Professor Miles S. Kersten (21, pages 86, 87, 88, 90). The writers have changed the form of presentation from that originally given but the conductivity values, which are dependent primarily on water content and dry density, have not been altered.

Figure 3: The Depth of Penetration part of the nomograph may be considered to consist of three components: (1) the portion above the α, μ, λ curves solves the equations for the fusion parameter μ and the thermal ratio α ; (2) the nest of curves represents the relationship between α, μ and the correction coefficient; (3) the portion to the right of the curves solves the equation for the depth of penetration X . Origin of the equations and α, μ, λ curves is presented in Appendix A.

Table of Notation: The symbols used on the nomograph are generally consistent with prevailing notation. There are two important exceptions, however: (1) "n" has been used for the surface transfer coefficient and (2) " v_0 ", which commonly refers to the mean annual temperature, is the number of degrees F by which the mean annual temperature exceeds the freezing point of soil moisture. For example, if the freezing point of soil moisture is taken equal to 31°F , then, $v_0 = (\text{mean annual temperature}) - 31^{\circ}$.

List of Equations: All equations given in this list are either well known or are derived in Appendix A. The nomograph is primarily a graphical solution of these seven equations.

Example of the Use of the Frost Penetration Nomograph: This example is discussed in some detail under the following section.

3-02. DIRECTIONS FOR THE USE OF THE NOMOGRAPH.

The following paragraphs present concise directions for the use of the frost penetration nomograph. These directions are based on the writers' present state of knowledge regarding the effect of important variables. Limitations and assumptions are given in the following section.

a. General. - Before the nomograph can be used the following terms must first be calculated or estimated.

(1) Climatic conditions of the locality; v_o , F , and t . These values may be determined directly from weather bureau records or estimated from contour maps. (For example, Reference 22, page 149).

(2) Type of Surface; Until further data are available, use $n = 0.9$ for all pavement surfaces. The basis for this assumption is discussed in Section 3-03-d.

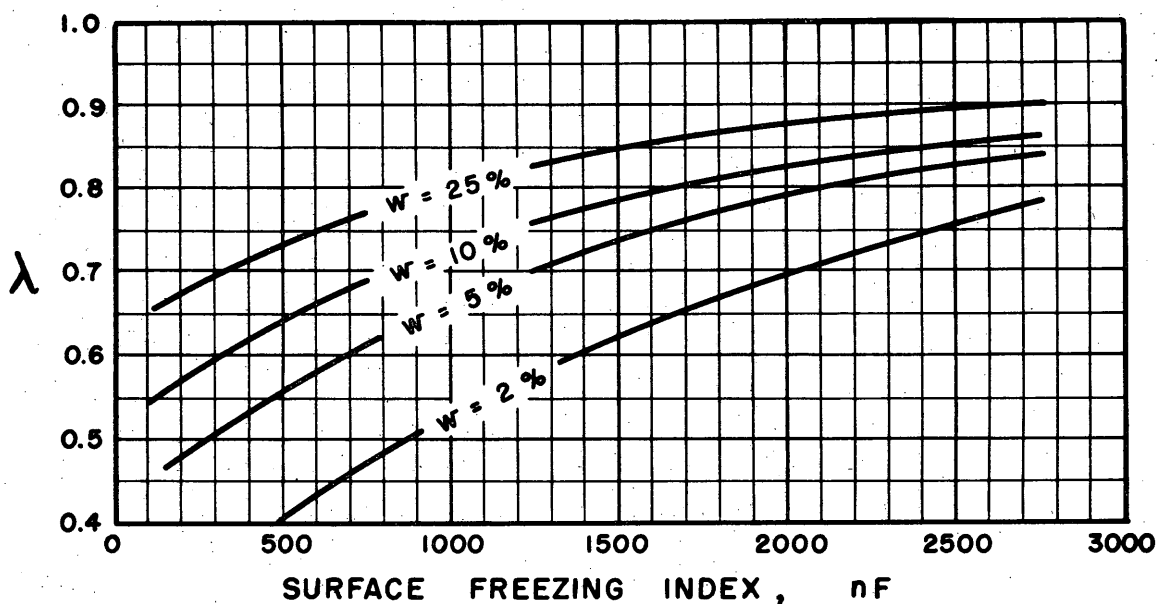
(3) Soil Properties : For each soil stratum within the depth subject to freezing; w , γ_o , and the soil type (whether predominantly sandy or clayey). A soil which is not distinctly either type may be treated as explained in Section 3-03-a.

b. Homogeneous Soil. - This case is represented by the "Example of the Use of the Frost Penetration Nomograph". One can enter, with a soft pencil, data for the case at hand in the blanks provided in the table. On the nomograph, an arrow pointing into a scale indicates a value which is entered. An arrow pointing away from the scale represents a number which is read out for use elsewhere. Each line connecting nomographic scales, has a sequence number and two arrows. The latter indicate the direction of travel required to obtain the answer for that step.

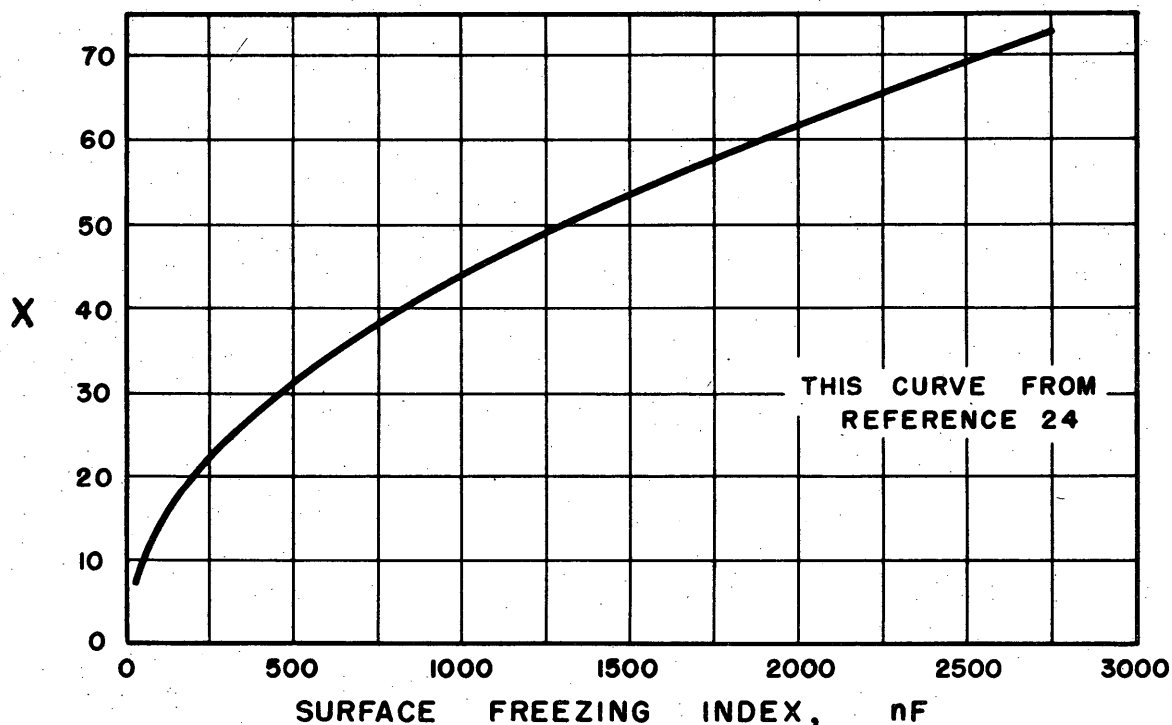
c. Stratified Soil. - The following steps for this solution are suggested:

Step 1 Determine the approximate depth of frost penetration X from Figure 9-B of this report.

Step 2 Determine the approximate correction factor λ from Figure 9-A of this report. Use a weighted average value of water content within the estimated depth of freezing X .



A. SEMI-EMPIRICAL CURVES FOR CORRECTION FACTOR λ



B. APPROXIMATE DEPTH OF FROST PENETRATION BELOW PAVEMENTS WITH GRANULAR BASE COURSES

**CURVES USED FOR DETERMINATIONS OF
THE DEPTH OF FROST PENETRATION**

FIGURE 9

Step 3 Determine L for each stratum from Figure 1 of the nomograph.

Step 4 Determine k for each stratum from Figure 2 of the nomograph.

Step 5 Compute the effective $\frac{K}{L}$ from the formula:

$$\begin{aligned} \left(\frac{L}{k}\right)_{\text{eff.}} = & \frac{2}{x^2} \left[\frac{d_o}{k_o} \left(\frac{L_o d_o}{2} + L_1 d_1 + \dots L_n d_n \right) \right. \\ & + \frac{d_1}{k_1} \left(\frac{L_1 d_1}{2} + L_2 d_2 + \dots L_n d_n \right) \\ & \left. + \dots + \frac{d_n}{k_n} \left(\frac{L_n d_n}{2} \right) \right] \end{aligned}$$

Step 6 Use any effective value for k and determine an effective L. Suggestion: use k = 1.0, then

$$L = \left(\frac{L}{k}\right)_{\text{eff.}} \text{ from Step 5.}$$

Step 7 Enter the portion of Figure 3 to the right of the

α, μ, λ curves on the frost penetration nomograph with values determined in Steps 2 and 6 to obtain a corrected depth of frost penetration X.

If X differs appreciably from that determined in Step 1, then:

Step 8 Adjust the value of λ from Step 2.

Step 9 Correct the value of $\left(\frac{L}{k}\right)_{\text{eff.}}$ from Step 5.

Step 10 Redetermine X as in Steps 6 and 7.

Repeat the above steps, if necessary.

d. Negative v_0 (Case where mean annual temperature is below the freezing point of soil moisture). - For the homogeneous case, proceed with steps 1, 5, 6, 7, and 8 of the "Example of the Use of the Frost Penetration Nomograph", then use $\lambda = 0.9$ to complete the computation with steps 13, 14 and 15. For the stratified case, use $\lambda = 0.9$ in Step 2 of Section 3-03-c.

3-03. LIMITATIONS, ASSUMPTIONS, AND OTHER SPECIAL CASES. - It is obvious that the accuracy of the nomograph for predicting the depth of frost penetration depends to a large extent on the dependability of the data used in the computation. This situation is not an unfamiliar one in soil engineering problems. If the soil properties of water content and dry density are at best only crude approximations or if the climatic conditions are not too well known then the nomograph should probably be disregarded. In this case, simplified procedures utilizing curves such as those in Figures 9-A and 9-B are indicated.

The simple curve of Figure 9-B which represents a first approximation, may be expected to yield predicted depths of frost penetration below pavements having granular base courses, differing not more than 50 per cent from the actual if the freezing index is greater than 500 degree days. A second stage approximation may be made by utilizing the semi-empirical curves of Figure 9-A to determine a value of the correction coefficient λ for use in the equation

$$X = 12\lambda \sqrt{\frac{48 \text{ knF}}{L}}$$

which can be solved by the portion of the nomograph to the right of the α, u, λ curves. The contractors feel that Figures 9-A and 9-B might very well be added to the nomograph at some future date.

The frost penetration nomograph is, of course, subject to all of the limitations and assumptions used to derive the rational formula. These are stated and discussed in Section 2-02-e. The importance of these assumptions as a whole can best be shown by statistical studies comparing predicted and actual depths of frost penetration. The reader is referred to Section 2-02-d-(2).

a. Limitation on k. - According to Professor Miles Kersten⁽²¹⁾, the thermal conductivity values from Figure 2 of the nomograph are good to ± 25 per cent.

If the soil under consideration is a well graded one which is neither truly sandy nor a silt or clay then the conductivity may be estimated by interpolation between the dashed and solid curves for the given water content and dry density. No specific rules can be offered at this time for the interpolation. To quote Professor Kersten, "I myself sometimes study the test results on the individual soils in Appendix 1⁽²¹⁾ to find a soil similar to the one for which k is desired."

b. Allowance for Frost Heave. - If freezing temperatures penetrate into a frost susceptible soil which is part of an open system, ice lenses may form and cause considerable heave of the ground surface. The maximum depth of frost penetration in this instance will be smaller than for a corresponding case with no ice segregation. In effect, the water content of the soil is increased as water flows upward to the zone of

freezing. This, in turn, increases the latent heat of fusion which is the most important soil characteristic affecting the depth of frost penetration. Since $X \sim \sqrt{\frac{1}{L}}$ it follows that the maximum depth of frost penetration will be smaller.

If the amount of frost heave due to lense growth can be reasonably estimated, an equivalent average water content, greater than the normal, and a corresponding dry density, smaller than the normal, may be determined for the soil in the frozen state. These values may then be used in the nomograph to predict the depth of frost penetration.

c. Thaw. - The nomograph at the present time is not directly applicable for use in predicting the rate at which thawing occurs during the spring months. The writers will study this case under a Corps of Engineers contract during the Fiscal Year 1953-1954.

Carlson and Kersten ⁽²³⁾ have shown that the depth of thawing below pavements in Alaska may be accurately predicted assuming $\lambda = 1.0$ in the formula

$$X = 12\lambda \sqrt{\frac{48knI}{L}}$$

where $n = 1.4$ and I represents the air thawing index.

d. Surface Transfer Coefficient, n . - The writers have selected for use at this time a value of n equal to 0.9 for all pavement surfaces. There can be little question, however, that n for bituminous concrete pavements is smaller than for Portland cement pavements because of the increased absorptivity for solar radiation in the former. The Frost Effects Laboratory ⁽¹⁶⁾ has, in fact, used values of n equal to 0.75 and 0.90, respectively, for the two pavement types. These values were

arrived at through an analysis of thermocouple measurements in the two types of pavements at three site locations. Nevertheless, results of recent statistical studies by the writers and the Frost Effects Laboratory using the modified Berggren formula, Section 2-02-d-(2), indicate that computed depths of freezing are generally too small when pavement freezing indices equal to 0.75 and 0.9 are used. Furthermore, there is less scattering of statistical results when the air freezing index is employed. Therefore, it is recommended that until further data and analyses are available, a value of n equal to 0.9 be used.

The surface freezing index for other surface types including trees, moss, grass, loam, etc., are undoubtedly considerably smaller than 0.9. The writers will not venture a recommendation at this time for these surface types. Some data are available in Reference (16).

The following factors are known to affect the surface temperature transfer phenomenon:

- (1) Ambient air temperature;
- (2) Surface connective heat transfer coefficient;
- (3) Solar radiation, direct and diffuse;
- (4) Absorptivity for Solar radiation;
- (5) Long-wave radiation (a function of cloudiness and humidity);
- (6) Emissivity and absorbtivity for long-wave radiation;
- (7) Precipitation.

The writers are currently investigating the factors listed above and expect to report on their importance as a part of researches for the Corps of Engineers during the period 1953-1954.

e. Limitations due to Stratification. - A procedure for applying the nomograph to layered soils has been given in Section 3-02-c. This technique should be restricted to cases which lie between the extremes of very small and very large differences in thermal properties from layer to layer. In other words, if only very slight variations occur in soil type, dry weight and water content, simple average soil constants may be determined and the nomograph used as detailed in Section 3-02-b. On the other hand, if the stratification occurs with extreme changes in properties, the procedure of Section 3-02-c will lead to only very approximate results, which, however, should be as reliable as those computed by any other technique now available.

More accurate determinations of this last case would require use of the more elaborate methods outlined under Section 2-01-f-(4), in which event, the nomograph would be of little use. If such procedures are employed, the number of depth nodes (the range of n) should be at least twice the number of distinct layers.

3-04. CONCLUSIONS AND RECOMMENDATIONS. - The nomograph as presented in the previous sections is felt to be a useful and rapid computational aid.

It is recommended for users of the nomograph that a record be kept on the $\lambda - \alpha - u$ curves of all values of λ obtained from actual depth measurements, whenever sufficient data are available for such checks. This permits the nomograph to be used in a semi-empirical fashion for more reliable predictions in a given locality.

It is further recommended that a table of n values for different surface types be included on the nomograph when more reliable information becomes available. In addition, it is suggested that curves similar to those shown in Figure 9 be added to the nomograph. These additions along with thaw considerations will further the objective of making the nomograph a complete entity for determining the depth of freezing and thawing.

PART IV. THAW-CONSOLIDATION PROBLEM

4-01. SYNOPSIS. -

In the following section, the physical nature of the so-called thaw-consolidation problem is discussed. This is followed by a mathematical formulation of the problem and a discussion of solution techniques.

It is apparent that a rigorous analytical solution of this problem is impossible without questionable simplifying assumptions. Numerical methods by hand or I.B.M. and hydraulic models offer the best possibilities for practical solutions. Important considerations bearing upon the solution of this problem are discussed in earlier sections, principally under Practical Methods of Computation (Section 2-01-f) and Depth-Time Curves (Section 2-02-f).

It is recommended that this problem continue to be explored in a limited way with additional I.B.M. and hand numerical solutions but that the bulk of the work be postponed until an hydraulic analog is constructed along the principles outlined in Section 2-01-f-(1).

4-02. PHYSICAL NATURE OF THE PROBLEM. -

If below-freezing temperatures penetrate into a frost-susceptible soil, significant ice segregation is likely to occur. In this state, the soil stratum contains an excessive amount of water. More specifically, the soil exists at a higher average void ratio and water content than it normally would under the weight of the overlying material.

In the spring, as warm weather approaches, thawing progresses from ground surface downward into the supersaturated soil. In effect, for each unit of depth the thawing releases a charge of water which was

stored in the soil in the form of ice lenses. This excess water must initially flow vertically upward, if a one-dimensional case is visualized, through the thawed soil to a drainage surface. The rate at which the dissipation occurs depends principally on the permeability of the overlying soil and the distance the water must travel to a drainage surface. After the soil stratum has completely thawed, the excess water is eventually entirely dissipated through a seepage diffusion process. The soil has adjusted, then to a stable void ratio under the weight of the overlying soil.

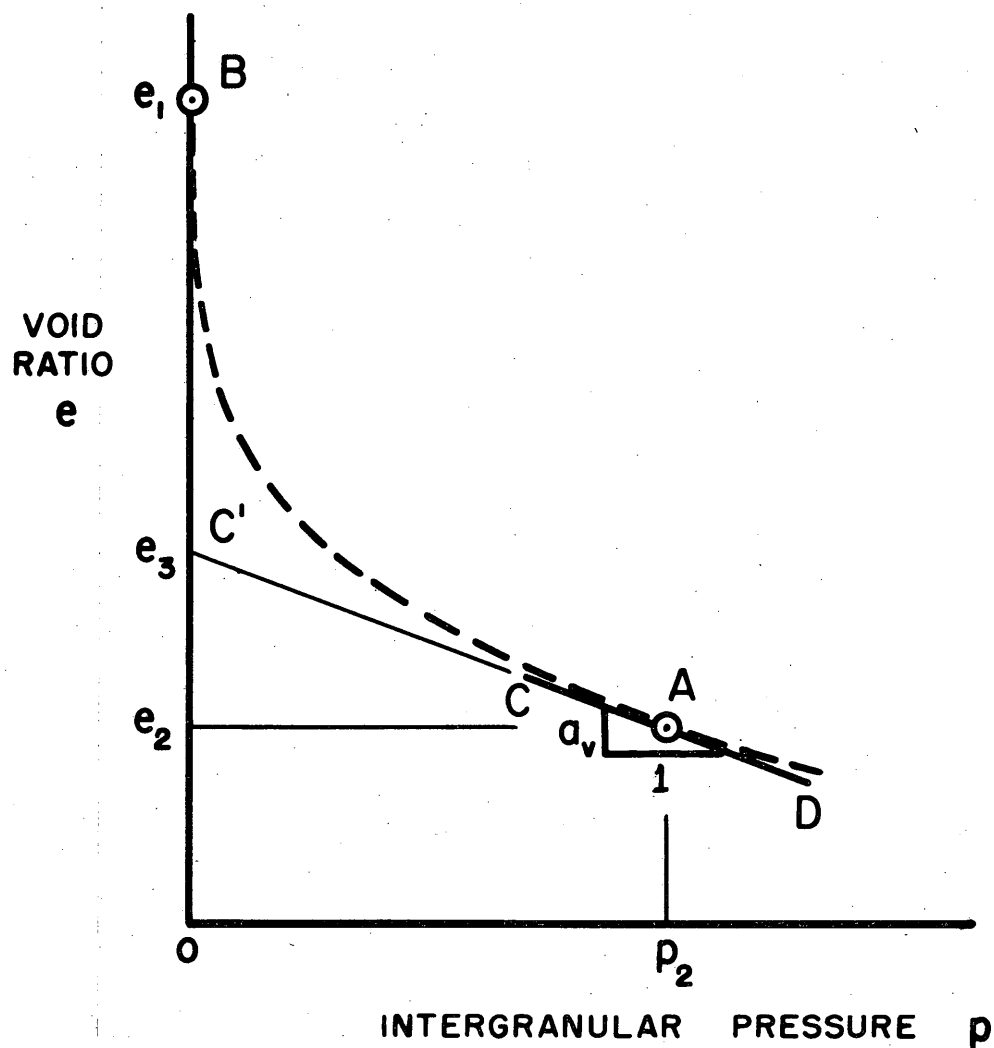
4-03. MATHEMATICAL FORMULATION. -

Before formulating the equations and boundary conditions applying to this problem it is important to investigate the pressure-void ratio relationship of an element of soil during the thawing process.

In the classical one-dimensional consolidation theory a straight line relationship between void ratio and pressure is assumed. It is apparent that for frost-heaves of appreciable magnitude the void ratio of the frozen soil will increase to a value far greater than that given by the usual straight line relationship extrapolated to zero pressure. The straight line relationship thus needs clarification.

Let e_2 and p_2 , Figure 10, represent the void ratio and intergranular pressure, respectively, in an element of soil before ice-segregation occurs. Point A thus represents an equilibrium state under the weight of the overlying soil. Line CD through point A represents a portion of the pressure-void ratio relationship of the element. Its local slope, $\frac{de}{dp}$, is thus the coefficient of compressibility a_v .

When significant ice segregation occurs, the void ratio in effect increases to some value e_1 . Most of the load is actually carried by the ice.



PRESSURE - VOID RATIO CURVE THAW - CONSOLIDATION

FIGURE 10

If suddenly thawed, the intergranular pressure would thus be very nearly zero as the element begins to consolidate. Assume, then, that point B represents the soil both in the frozen state and immediately after thaw.

After thawing, the element follows some path from B to A as it consolidates under the weight of the overlying soil. The path is probably given by a curve similar to that indicated in Figure 10 by the heavy dashed line.

As a first approximation, it is reasonable to assume that the element follows the path BC'A. Under this assumption the physical seepage and consolidation process may be described as follows. On first thawing the element has a void ratio e_1 , and a hydrostatic excess water pressure equal to p_2 . The excess water pressure remains constant in the element until enough stored water has been dissipated to lower the void ratio to e_3 . Flow occurs throughout this period under a constant excess water pressure p_2 since the intergranular pressure is zero. In consolidating from e_3 to e_2 the soil follows C'A (with $a_v = \text{constant}$, determined at A) as the excess pressure is dissipated according to the unsteady flow diffusion process, governed by the fundamental equations:

$$Q = -k \frac{\partial H}{\partial x} \quad \dots(4-1)$$

$$\frac{\partial s}{\partial t} + \frac{\partial Q}{\partial x} = 0 \quad \dots(4-2)$$

$$\Delta s = \frac{\Delta e}{1+e} \quad \dots(4-3)$$

$$\Delta u = \gamma_w \Delta H = \frac{\Delta e}{a_v} \quad \dots(4-4)$$

which may be combined into the classical diffusion equation for the

hydrostatic excess pressure u , as: $c_v \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad \dots(4-5)$

where $c_v = \frac{k(1+e)}{\gamma_w a_v}$

4-04. SOLUTION TECHNIQUES. -

Even with the simplifying assumptions as stated above, the thaw-consolidation problem as formulated in the previous section is not amenable to closed form analytical solution.

This difficulty suggests the application of methods similar to those outlined in Section 2-01-f; these are detailed in the sections to follow.

a. Hydraulic Model. - The use of an hydraulic model represents one of the most efficient methods of solving the thaw-consolidation problem. In addition, the model presents a graphic picture of the state of consolidation within the soil stratum at any time. The relationship of an hydraulic model to the thermal diffusion problem with latent heat is described in Section 2-01-f-(1). Its relationship to the consolidation process is presented in References 3 and 4. Only the direct application of the model to the thaw-consolidation process will be discussed here.

A simplified hydraulic model for the one dimensional thaw-consolidation process is shown in Figure 11. This particular model is based on the assumption that the soil is homogeneous and that each soil element follows a path similar to line BC'A, Figure 10, as it consolidates following thaw. It is further assumed for this discussion that the coefficient of permeability k remains constant during the consolidation process.

As a result of ice-segregation in the frozen portion of the soil an excess amount of water is stored in each layer. This excess water is represented by the volume of water in the reservoir and corresponding standpipe. The water volume in the reservoir represents the void ratio change $e_1 - e_3$, Figure 10, while that in the standpipe corresponds to

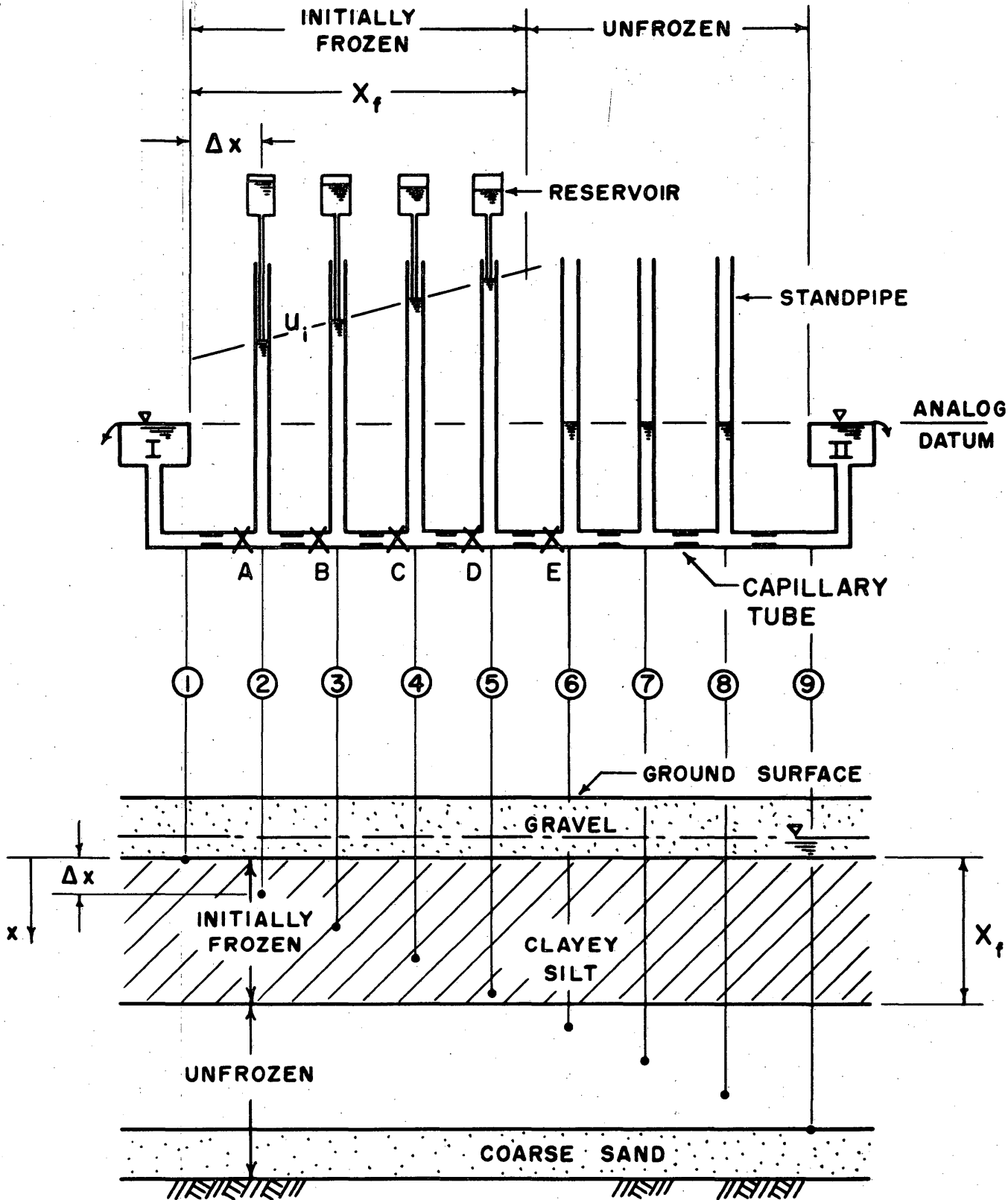
$e_3 - e_2$. After thaw, the initial excess pressure u_1 in each soil element is equal to the effective weight of the overlying soil. This assumption fixes the initial water level in the standpipes.

When thawing penetrates a distance Δx , the excess water stored in the first layer of soil is released. In essence, valve A is opened and steady flow commences toward the adjacent tank I under a head differential equal to u_{i2} . As soon as the reservoir is empty unsteady flow from the standpipe occurs under continuously decreasing head.

Successive valves are opened as each layer thaws. When the depth of thaw equals X_f , the soil within the unfrozen lower portion of the stratum takes part in the diffusion process and swells as water flows vertically downward toward the bottom drainage surface. On the model this condition is initiated by opening the last valve E.

Any specific thaw-consolidation problem may be treated using two separate hydraulic models. The first model is set up for the thermal problem of thaw while the second, similar to that in Figure 11, represents the accompanying consolidation process. The first model indicates the time when each valve in the second should be opened.

It is theoretically possible to construct the model such that it can represent complex pressure-void ratio functions during thaw, variations in permeability with void ratio, and other features of stratification and non-homogeneity. These refinements may be incorporated when knowledge regarding the actual thaw-consolidation characteristics of soil has been advanced. Laboratory and full scale field tests are needed for this purpose.



**SIMPLIFIED HYDRAULIC MODEL
FOR THAW-CONSOLIDATION**

FIGURE II

b. Numerical Methods. - Numerical methods of solution applicable to the thaw-consolidation problem parallel the methods discussed in Sections 2-01-d, 2-01-f-(3) and 2-01-f-(4). They can be divided into two types, hand computations and machine computations.

(1) Hand Computations. - The technique of solving a problem of this type is best illustrated by an example. Such an example is presented in Appendix C.

(2) Machine Computations. - The use of automatic digital computing equipment such as the I.B.M. Card Programmed Calculator represents a rapid but relatively costly method of obtaining solutions to the thaw-consolidation problem. Techniques very similar to those described in Section 2-01-f-(3) would be used.

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Appendix A

DERIVATION OF A RATIONAL FORMULA FOR THE PREDICTION OF FROST PENETRATION

A-01. BASIC ASSUMPTIONS. -

We seek a solution to the thermal problem of a semi-infinite soil mass of uniform properties and at a uniform initial temperature subjected to changes of temperature at the surface, which are assumed uniform over the surface extent to yield a one-dimensional problem.

Figure 4 illustrates the nomenclature and significant variables in this situation. Further assumptions are best listed in the form of specific conditions as follows:

Condition I - AT THE GROUND SURFACE

It is assumed that the surface temperature is suddenly changed from an initial temperature v_0 above freezing to a temperature v_s below freezing. This temperature value is then maintained constant and uniform over the entire surface.

Condition II - IN THE FROZEN SOIL

It is assumed that the diffusion equation:

$$a_f \nabla^2 v_f = \frac{\partial v_f}{\partial t}$$

with $a_f \equiv k_f/c_f$ measuring the diffusivity of the frozen soil is satisfied throughout the frozen soil mass, subject to the surface temperature condition (I) and the latent heat condition (III) at its boundaries.

Condition III - AT THE MOVING FROST INTERFACE

It is assumed that at the interface between the frozen soil (above) and the unfrozen soil (below) the temperature remains constant

at the freezing point of the soil moisture. It is further assumed that the heat flow upward just above the interface in the frozen soil is equal to the sum of the heat flow just below the interface in the unfrozen soil plus the heat flow due to the removal of the latent heat of fusion of the soil moisture as it freezes.

Condition IV - IN THE UNFROZEN SOIL

It is assumed that the diffusion equation:

$$a_u \nabla^2 v_u = \frac{\partial v_u}{\partial t}$$

with $a_u = k_u / C_u$ measuring the diffusivity of the unfrozen soil, is satisfied throughout the unfrozen region, subject to the latent heat condition (III) at the frost interface (the upper boundary of the unfrozen soil) and to the lower boundary condition that, at all times, the temperature approaches the initial temperature for sufficiently great depths.

A-02. GENERAL SOLUTION. -

Following the notation of Figure 4, the solution for v_f , which satisfies conditions I and II may be written:

$$v_f = -v_s + A \operatorname{erf} \left(\frac{X}{2 \sqrt{a_f t}} \right) \quad \dots(A-1)$$

while the solution for v_u satisfying condition IV may be written:

$$v_u = v_o + B \left[1 - \operatorname{erf} \left(\frac{X}{2 \sqrt{a_u t}} \right) \right] \quad \dots(A-2)$$

At the interface ($x = X$), in order to satisfy condition III, where the freezing point is taken arbitrarily as zero degrees, it is necessary that

$$v_f = v_u = 0 \quad \dots(A-3)$$

from which:

$$A \operatorname{erf} \left(\frac{X}{2 \sqrt{a_f t}} \right) = v_s \quad \dots (A-4)$$

$$B \left[1 - \operatorname{erf} \left(\frac{X}{2 \sqrt{a_u t}} \right) \right] = -v_o \quad \dots (A-5)$$

But since these last conditions must be satisfied for all values of time, then

$$X/\sqrt{t} = \gamma = \text{constant} \quad \dots (A-6)$$

or

$$X = \gamma \sqrt{t} \quad \dots (A-7)$$

Thus the constant γ depends upon v_s , v_o and the thermal coefficients.

The manner of this dependence may be found by considering the latent heat requirement of condition III. This thermal continuity condition relates the temperature gradients each side of the interface to the rate of movement of the interface in the form:

$$k_f \frac{\partial v_f}{\partial x} - k_u \frac{\partial v_u}{\partial x} = L \frac{dX}{dt} \quad \dots (A-8)$$

which becomes, upon substituting the expressions (A-4), (A-5) and (A-7), performing appropriate differentiation, and simplifying:

$$\frac{v_s k_f}{\sqrt{\pi a_f}} \cdot \frac{e^{-\gamma^2/4a_f}}{\operatorname{erf}(\gamma/2\sqrt{a_f})} - \frac{v_o k_u}{\sqrt{\pi a_u}} \cdot \frac{e^{-\gamma^2/4a_u}}{[1 - \operatorname{erf}(\gamma/2\sqrt{a_u})]} = \frac{L}{2} \gamma \quad \dots (A-9)$$

Furthermore, by making the substitutions:

$$\alpha \equiv v_o C_u / v_s C_f \quad \dots (A-10)$$

$$\delta \equiv \sqrt{a_f/a_u} = \sqrt{k_f C_u / k_u C_f} \quad \dots (A-11)$$

$$\mu \equiv v_s C_f / L \quad \dots (A-12)$$

$$Z \equiv \delta / 2 \sqrt{a_f} \quad \dots (A-13)$$

it is possible to rewrite Equation (A-9) in a simplified non-dimensional form, namely:

$$\mu \left[\frac{e^{-Z^2}}{\operatorname{erf} Z} - \frac{\alpha}{\delta} \frac{e^{-\delta^2 Z^2}}{(1 - \operatorname{erf} \delta Z)} \right] = \sqrt{\pi} Z \quad \dots (A-14)$$

If all the terms in Z are then carried to the right hand side, one has a direct relation between μ and α , δ , and Z in the form:

$$\mu = \mu(\alpha, \delta, Z) = \sqrt{\pi} Z / \left[\frac{e^{-Z^2}}{\operatorname{erf} Z} - \frac{\alpha}{\delta} \frac{e^{-\delta^2 Z^2}}{(1 - \operatorname{erf} \delta Z)} \right] \quad \dots (A-15)$$

This last may then be inverted to obtain a transcendental relation for Z in terms of the parameters α , δ , and μ , which may be solved graphically to give, in symbolic form:

$$Z = Z(\alpha, \delta, \mu) \quad \dots (A-16)$$

Thus using Equation (A-13), one may obtain:

$$\delta = 2 \sqrt{a_f} \cdot Z \quad \dots (A-17)$$

and, finally, for the frost penetration depth:

$$x = \delta \sqrt{t} = 2 Z \sqrt{a_f t} \quad \dots (A-18)$$

It is found convenient to rearrange this last expression so that the radical is expressed in terms of μ in the form:

$$x = \left[\frac{2 Z}{\sqrt{2 \mu}} \right] \sqrt{2 \mu} \sqrt{a_f t} \quad \dots (A-19)$$

By defining the new dimensionless correction coefficient λ as

$$\lambda \equiv \sqrt{\frac{2Z^2}{\mu}} = \lambda(\alpha, \delta, \mu) \quad \dots(A-20)$$

and by substituting under the radical in Equation (A-19) for the physical variables, one finally obtains for X, the general expression:

$$X = \lambda \sqrt{\frac{48k_f v_s t}{L}} \quad \dots(A-21)$$

for time measured in days. The correction coefficient λ is, again, a function of the three dimensionless parameters α, δ, μ .

A-03. PHYSICAL SIGNIFICANCE OF TERMS AND PARAMETERS. -

a. Surface Temperature Change (v_s). - In order to apply Equation (A-21) to actual cases in which the surface temperature does not change suddenly as assumed in this derivation, the average temperature below freezing during the freezing period is taken as v_s . Thus, in terms of the freezing index F (in degree-days) and the freezing period t (in days), the surface temperature v_s may be written:

$$v_s = F/t \quad \dots(A-22)$$

or, solving for F,

$$F = v_s t \quad \dots(A-23)$$

The effects of the actual shape of the freezing index vs. time curve -- as compared with the assumed linear shape -- are not significant so long as the freezing period is not too short nor the freezing index too small (i.e., cases in which the depth of freeze would normally be fairly small.) This problem is treated in Section 2-02-f.

b. Thermal Ratio (α). - The thermal ratio α , defined as

$$\alpha = v_o C_u / v_s C_f \quad \dots(A-24)$$

measures the ratio of heat stored initially in the unfrozen soil to the heat loss in the frozen soil. If it may be assumed, as with the existing formulas, that the difference between the volumetric heats C_u and C_f is not usually significant, the thermal ratio α may be written

$$\alpha = v_o/v_s \quad \dots(A-25)$$

which is the ratio of the initial ground temperature (or mean annual temperature) above the freezing point to the average surface temperature below freezing during the freezing period.

In terms of the freezing index F and the freezing period t , this may then be written

$$\alpha = v_o t / F \quad \dots(A-26)$$

c. Diffusivity Ratio (δ). - The (root) diffusivity ratio δ , defined as

$$\delta = \sqrt{a_f/a_u} \quad \dots(A-27)$$

measures the relative values of diffusivity $a = k/C$ in the frozen and unfrozen soils. It is clear that for most soils of low moisture content δ is approximately unity. Tabulated herewith are typical values of δ for representative soil types and moisture contents.

TABLE A - I

TYPICAL VALUES OF THE DIFFUSIVITY RATIO

$$\delta = \sqrt{k_f C_u / k_u C_f}$$

Assumed Dry Density = 110 lbs/ft³

Moisture Content %	Values of δ	
	Silt or Clay	Sand
0	1.00	1.00
5	1.07	0.92
10	1.15	1.14
15	1.22	1.31
20	1.30	1.50

Source of Data for k and C: "Final Report, Laboratory Research for the Determination of the Thermal Properties of Soils" C. of E., St. Paul District, June 1949.

The effect of variations in δ may be found directly from the λ - α - μ graph, Figure 5, by making use of an equivalent α -value given by the expression:

$$\alpha_{\delta} = \frac{\alpha}{\delta} \left[\frac{1 - \operatorname{erf} \lambda \sqrt{\mu/2}}{1 - \operatorname{erf} \delta \lambda \sqrt{\mu/2}} \right] e^{-(\delta^2 - 1) \frac{\lambda^2 \mu}{2}} \dots (A-28)$$

which can be approximated by the empirical equation:

$$\alpha_{\delta} \approx \frac{\alpha}{\delta} \left[1 + \frac{\delta - 1}{2 \sqrt{1 + \alpha}} \right] \dots (A-29)$$

A comparison of exact values of λ determined from Equations (A-15) and (A-20) with estimated values using Figure 5 and Equation (A-29) shows only negligible differences for all practical values of δ .

The following table indicates the relationship between α , δ and α_δ as given by Equation (A-29).

TABLE A - II

EQUIVALENT THERMAL RATIO (α_δ)

$$\text{Values of } \alpha_\delta = \frac{\alpha}{\delta} \left[1 + \frac{\delta - 1}{2\sqrt{1 + \alpha}} \right]$$

VALUES OF α	VALUES OF δ						
	0.9	1.0	1.1	1.2	1.3	1.4	1.5
0	0	0	0	0	0	0	0
1	1.07	1.00	0.94	0.89	0.85	0.82	0.79
2	2.16	2.00	1.87	1.76	1.67	1.59	1.53
3	3.25	3.00	2.80	2.62	2.48	2.36	2.25
4	4.35	4.00	3.72	3.47	3.28	3.11	2.97

Since it has been demonstrated that a 1% error in water content produces only about a 1.5% change in the value of δ , the above tabulation in conjunction with Figure 5 would indicate less than a 1% change in the computed depth of penetration due to this variation. Moreover, typical soils have δ values of the order of 1.15, which would indicate an effective value of α_δ which is roughly 15% smaller than that computed assuming $\delta = 1.0$, yet this effect can produce a variation in λ and therefore a change in predicted depth of less than 5% for typical thermal conditions.

Since inclusion of a term involving δ would increase the complexity of representation of the formula in graphical form, and would further increase the tendency to predict depths of penetration which are

too deep, it seems reasonable to base at least preliminary calculations on the assumption that $\delta = 1.0$.

Consistent, then, with the previous assumption that $C = C_u = C_f$ and $\delta = \sqrt{\left(\frac{k_f}{k_u}\right) \left(\frac{C_u}{C_f}\right)} = 1.0$, there would follow then the necessary condition that $k = k_f = k_u$. In practical applications, this would suggest the use of average values of C and k .

d. Fusion Parameter (μ). - The fusion parameter μ , defined as

$$\mu = v_s C_f / L \quad \dots(A-30)$$

measures the heat removed in the frozen soil (below the freezing point) compared to the latent heat of the soil moisture. This characteristic may be written in terms of F , t , and C in the form

$$\mu = FC/Lt \quad \dots(A-31)$$

When $\mu = 0$, the only significant soil properties affecting the depth of frost penetration are the latent heat L , and the thermal conductivity, k_f ; on the other hand, as μ becomes large, the stored heat in the soil volume becomes proportionately more significant.

e. Correction Coefficient (λ). - If Equation (A-22) is substituted into Equation (A-21) we obtain the more familiar expression

$$x = \lambda \sqrt{\frac{48kF}{L}} \quad \dots(A-32)$$

and λ , therefore, becomes a correction coefficient applied to the calculated depth of penetration due to latent heat only.

The value of λ may be found from Figure 5 which shows the relationship between α , μ and λ for the case of $\delta = 1$. Data for this figure were obtained by assuming values of z and solving Equation (A-15) for μ . Once μ was obtained λ followed directly from Equation (A-20).

A-04. COMPARISON WITH BERGGREN FORMULA. -

As presented in Section 2-02-b. Shannon ⁽¹⁹⁾ has written the Berggren formula ⁽¹⁸⁾ in the form:

$$X = 2B \sqrt{a_f t} \quad \dots(A-33)$$

By comparison with Equation (A-18) of Section A-02, it is clear that the term B in Equation (A-33) is identical with the term Z in Equation (A-18). Thus the only significant difference between the older Berggren formula and the present rational formula lies in the inversion of the nuclear radical term to permit expressing the correction factor λ to the depth of frost penetration that would occur if only latent heat were present, rather than the correction term $B \equiv Z$ applied to the depth occurring if only volumetric heat is significant. Since for all practical problems the effects of L are large compared to those of C (i.e., the μ values are low), this seems to be the more useful form for the same fundamental solution.

For the above reasons, it would seem fitting to refer to the rational formula proposed here as the "modified Berggren formula".

APPENDIX B

TABLES OF I.B.M. SOLUTIONS

B-01. GENERAL. -

As outlined in Section 2-01-f-(3), part of this investigation was concerned with obtaining precise numerical solutions to the difference equations which represent the thermal transients in the ground during the freezing and thawing processes. These equations are presented and explained in the section cited.

Solutions to these equations were run on the I.B.M. Card Programmed Calculator of the M.I.T. Statistical Services Division, with values and conditions indicated in the following tabulation.

B-02. INDEX TO SOLUTIONS. -

TABLE	SURFACE TEMPERATURE VARIATION	PARAMETERS		RANGE		COEFF.
		μ	α	n	k	β
B-I	Step	0.5	0+*	8	24	0.25
B-II	Step	0.5	1.0	8	24	0.25
B-III	Step	0.5	2.0	8	24	0.25
B-IV	Step	0.5	3.0	8	24	0.25
B-V	Step	0.5	4.0	8	24	0.25
B-VI	Step	0.5	5.0	8	24	0.25
B-VII	Step	1.0	0+*	8	24	0.25
B-VIII	Step	1.0	1.0	8	24	0.25
B-IX	Step	1.0	2.0	8	24	0.25
B-X	Step	1.0	3.0	8	24	0.25
B-XI	Step	1.0	4.0	8	24	0.25
B-XII	Step	1.0	5.0	8	24	0.25
B-XIII	Cyclic	1.0	0	7	108	0.40
B-XIV	Cyclic	1.0	1.0	7	108	0.40

*Temperature initially at the freezing point, soil unfrozen.

B-03. TABULATED VALUES. -

The tabular entries in the tables give the value of $e_{n,k}$ corresponding to each value of n and k. These values for n = 0 correspond

to the surface temperature variation, while the remaining values represent subsurface temperatures. The transformation of these dimensionless values, together with the time measure (k) and depth measure (n), are explained in Section 2-01-f-(3).

TABLE B-I
 $\mu = 0.5$ $\alpha = 0+$

k VALUES	n VALUES								
	0	1	2	3	4	5	6	7	8
0	-0.50	0	0	0	0	0	0	0	0
1	-0.50	0	0	0	0	0	0	0	0
2	-0.50	0	0	0	0	0	0	0	0
3	-0.50	0	0	0	0	0	0	0	0
4	-0.50	0	0	0	0	0	0	0	0
5	-0.50	0	0	0	0	0	0	0	0
6	-0.50	0	0	0	0	0	0	0	0
7	-0.50	0	0	0	0	0	0	0	0
8	-0.50	0	0	0	0	0	0	0	0
9	-0.50	-0.12	0	0	0	0	0	0	0
10	-0.50	-0.19	0	0	0	0	0	0	0
11	-0.50	-0.22	0	0	0	0	0	0	0
12	-0.50	-0.23	0	0	0	0	0	0	0
13	-0.50	-0.24	0	0	0	0	0	0	0
14	-0.50	-0.25	0	0	0	0	0	0	0
15	-0.50	-0.25	0	0	0	0	0	0	0
16	-0.50	-0.25	0	0	0	0	0	0	0
17	-0.50	-0.25	0	0	0	0	0	0	0
18	-0.50	-0.25	0	0	0	0	0	0	0
19	-0.50	-0.25	0	0	0	0	0	0	0
20	-0.50	-0.25	0	0	0	0	0	0	0
21	-0.50	-0.25	0	0	0	0	0	0	0
22	-0.50	-0.25	0	0	0	0	0	0	0
23	-0.50	-0.25	0	0	0	0	0	0	0
24	-0.50	-0.25	0	0	0	0	0	0	0

TABLE B-1

TABLE B-II
 $\mu = 0.5$ $\alpha = 1.0$

k VALUES	n VALUES								
	0	1	2	3	4	5	6	7	8
0	-0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50
1	-0.50	0.25	0.50	0.50	0.50	0.50	0.50	0.50	0.50
2	-0.50	0.12	0.44	0.50	0.50	0.50	0.50	0.50	0.50
3	-0.50	0.05	0.38	0.48	0.50	0.50	0.50	0.50	0.50
4	-0.50	0	0.32	0.48	0.50	0.50	0.50	0.50	0.50
5	-0.50	0	0.28	0.46	0.49	0.50	0.50	0.50	0.50
6	-0.50	0	0.26	0.46	0.49	0.50	0.50	0.50	0.50
7	-0.50	0	0.24	0.46	0.48	0.49	0.50	0.50	0.50
8	-0.50	0	0.24	0.45	0.48	0.49	0.50	0.50	0.50
9	-0.50	0	0.23	0.45	0.48	0.49	0.50	0.50	0.50
10	-0.50	0	0.23	0.45	0.47	0.49	0.50	0.50	0.50
11	-0.50	0	0.23	0.45	0.47	0.49	0.50	0.50	0.50
12	-0.50	0	0.22	0.45	0.47	0.48	0.49	0.50	0.50
13	-0.50	0	0.22	0.44	0.47	0.48	0.49	0.50	0.50
14	-0.50	0	0.22	0.44	0.46	0.48	0.49	0.50	0.50
15	-0.50	0	0.22	0.44	0.46	0.48	0.49	0.50	0.50
16	-0.50	0	0.22	0.44	0.46	0.48	0.49	0.50	0.50
17	-0.50	0	0.22	0.44	0.46	0.48	0.49	0.49	0.50
18	-0.50	0	0.22	0.44	0.46	0.48	0.49	0.49	0.50
19	-0.50	0	0.22	0.44	0.46	0.47	0.48	0.49	0.50
20	-0.50	-.06	0.22	0.44	0.46	0.47	0.48	0.49	0.50
21	-0.50	-.10	0.20	0.44	0.46	0.47	0.48	0.49	0.50
22	-0.50	-.12	0.19	0.43	0.46	0.47	0.48	0.49	0.50
23	-0.50	-.14	0.17	0.43	0.45	0.47	0.48	0.49	0.50
24	-0.50	-.15	0.16	0.42	0.45	0.47	0.48	0.49	0.50

TABLE B-II

TABLE B-III
 $\mu = 0.5$ $\alpha = 2.0$

k VALUES	n VALUES								
	0	1	2	3	4	5	6	7	8
0									
1									
2									
3									
4									
5									
6									
7									
8									
9									
10									
11									
12									
13									
14									
15									
16									
17									
18									
19									
20									
21									
22									
23									
24									

SOLUTION FOUND TO BE DEFECTIVE

TABLE B-III

TABLE B-IV
 $\mu = 0.5$ $\alpha = 3.0$

k VALUES	n VALUES								
	0	1	2	3	4	5	6	7	8
0	-0.50	1.50	1.50	1.50	1.50	1.50	1.50	1.50	1.50
1	-0.50	1.00	1.50	1.50	1.50	1.50	1.50	1.50	1.50
2	-0.50	0.75	1.37	1.50	1.50	1.50	1.50	1.50	1.50
3	-0.50	0.59	1.25	1.46	1.50	1.50	1.50	1.50	1.50
4	-0.50	0.48	1.14	1.45	1.49	1.50	1.50	1.50	1.50
5	-0.50	0.40	1.05	1.43	1.48	1.49	1.50	1.50	1.50
6	-0.50	0.34	0.98	1.41	1.47	1.49	1.49	1.50	1.50
7	-0.50	0.29	0.93	1.40	1.46	1.49	1.49	1.49	1.50
8	-0.50	0.25	0.89	1.39	1.45	1.48	1.49	1.49	1.50
9	-0.50	0.22	0.86	1.38	1.44	1.48	1.49	1.49	1.50
10	-0.50	0.20	0.83	1.37	1.44	1.47	1.49	1.49	1.50
11	-0.50	0.18	0.81	1.37	1.43	1.47	1.49	1.49	1.50
12	-0.50	0.17	0.79	1.36	1.42	1.46	1.48	1.49	1.50
13	-0.50	0.16	0.78	1.36	1.42	1.46	1.48	1.49	1.50
14	-0.50	0.15	0.77	1.35	1.41	1.45	1.48	1.49	1.50
15	-0.50	0.14	0.76	1.35	1.41	1.45	1.47	1.49	1.50
16	-0.50	0.14	0.75	1.34	1.40	1.44	1.47	1.48	1.50
17	-0.50	0.13	0.75	1.34	1.40	1.44	1.47	1.48	1.50
18	-0.50	0.13	0.74	1.33	1.39	1.44	1.46	1.48	1.50
19	-0.50	0.12	0.74	1.33	1.39	1.43	1.46	1.48	1.50
20	-0.50	0.12	0.73	1.33	1.39	1.43	1.46	1.48	1.50
21	-0.50	0.12	0.73	1.32	1.38	1.43	1.46	1.48	1.50
22	-0.50	0.12	0.73	1.32	1.38	1.42	1.45	1.47	1.50
23	-0.50	0.12	0.72	1.32	1.37	1.42	1.45	1.47	1.50
24	-0.50	0.11	0.72	1.32	1.37	1.42	1.45	1.47	1.50

TABLE B-IV

TABLE B-V
 $\mu = 0.5$ $\alpha = 4.0$

k VALUES	n VALUES								
	0	1	2	3	4	5	6	7	8
0	-0.50	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00
1	-0.50	1.37	2.00	2.00	2.00	2.00	2.00	2.00	2.00
2	-0.50	1.06	1.84	2.00	2.00	2.00	2.00	2.00	2.00
3	-0.50	0.87	1.68	1.96	2.00	2.00	2.00	2.00	2.00
4	-0.50	0.73	1.55	1.94	1.99	2.00	2.00	2.00	2.00
5	-0.50	0.63	1.44	1.91	1.98	1.99	2.00	2.00	2.00
6	-0.50	0.55	1.35	1.89	1.96	1.99	1.99	2.00	2.00
7	-0.50	0.49	1.29	1.88	1.95	1.98	1.99	1.99	2.00
8	-0.50	0.44	1.23	1.86	1.94	1.98	1.99	1.99	2.00
9	-0.50	0.40	1.19	1.85	1.93	1.97	1.99	1.99	2.00
10	-0.50	0.38	1.16	1.84	1.92	1.97	1.99	1.99	2.00
11	-0.50	0.35	1.13	1.83	1.91	1.96	1.98	1.99	2.00
12	-0.50	0.34	1.11	1.83	1.91	1.95	1.98	1.99	2.00
13	-0.50	0.32	1.10	1.82	1.90	1.95	1.98	1.99	2.00
14	-0.50	0.31	1.08	1.81	1.89	1.94	1.97	1.99	2.00
15	-0.50	0.30	1.07	1.81	1.88	1.94	1.97	1.98	2.00
16	-0.50	0.30	1.06	1.80	1.88	1.93	1.96	1.98	2.00
17	-0.50	0.29	1.05	1.80	1.87	1.93	1.96	1.98	2.00
18	-0.50	0.28	1.05	1.79	1.87	1.92	1.96	1.98	2.00
19	-0.50	0.28	1.04	1.79	1.86	1.92	1.95	1.98	2.00
20	-0.50	0.28	1.04	1.78	1.86	1.91	1.95	1.97	2.00
21	-0.50	0.27	1.03	1.78	1.85	1.91	1.95	1.97	2.00
22	-0.50	0.27	1.03	1.78	1.85	1.90	1.94	1.97	2.00
23	-0.50	0.27	1.03	1.77	1.84	1.90	1.94	1.97	2.00
24	-0.50	0.27	1.02	1.77	1.84	1.90	1.94	1.96	2.00

TABLE B-V

TABLE B-VI
 $\mu = 0.5$ $\alpha = 5.0$

k VALUES	n VALUES								
	0	1	2	3	4	5	6	7	8
0	-0.50	2.50	2.50	2.50	2.50	2.50	2.50	2.50	2.50
1	-0.50	1.75	2.50	2.50	2.50	2.50	2.50	2.50	2.50
2	-0.50	1.37	2.31	2.50	2.50	2.50	2.50	2.50	2.50
3	-0.50	1.14	2.12	2.45	2.50	2.50	2.50	2.50	2.50
4	-0.50	0.98	1.96	2.42	2.48	2.50	2.50	2.50	2.50
5	-0.50	0.85	1.83	2.39	2.47	2.49	2.50	2.50	2.50
6	-0.50	0.76	1.72	2.37	2.46	2.49	2.49	2.50	2.50
7	-0.50	0.69	1.64	2.35	2.44	2.48	2.49	2.49	2.50
8	-0.50	0.63	1.58	2.34	2.43	2.47	2.49	2.49	2.50
9	-0.50	0.59	1.53	2.32	2.42	2.47	2.49	2.49	2.50
10	-0.50	0.55	1.49	2.31	2.41	2.46	2.48	2.49	2.50
11	-0.50	0.22	1.46	2.30	2.40	2.45	2.48	2.49	2.50
12	-0.50	0.50	1.44	2.29	2.39	2.45	2.48	2.49	2.50
13	-0.50	0.49	1.42	2.29	2.38	2.44	2.47	2.49	2.50
14	-0.50	0.47	1.40	2.28	2.37	2.43	2.47	2.48	2.50
15	-0.50	0.46	1.39	2.27	2.36	2.43	2.46	2.48	2.50
16	-0.50	0.45	1.38	2.26	2.36	2.42	2.46	2.48	2.50
17	-0.50	0.45	1.37	2.26	2.35	2.41	2.45	2.48	2.50
18	-0.50	0.44	1.36	2.25	2.34	2.41	2.45	2.47	2.50
19	-0.50	0.44	1.35	2.25	2.34	2.40	2.44	2.47	2.50
20	-0.50	0.43	1.35	2.24	2.33	2.40	2.44	2.47	2.50
21	-0.50	0.43	1.34	2.24	2.32	2.39	2.44	2.47	2.50
22	-0.50	0.42	1.34	2.23	2.32	2.39	2.43	2.46	2.50
23	-0.50	0.42	1.33	2.23	2.31	2.38	2.43	2.46	2.50
24	-0.50	0.42	1.33	2.23	2.31	2.38	2.42	2.46	2.50

TABLE B-VI

TABLE B-VII
 $\mu = 1.0$ $\alpha = 0+$

k VALUES	n VALUES								
	0	1	2	3	4	5	6	7	8
0	-1.00	0	0	0	0	0	0	0	0
1	-1.00	0	0	0	0	0	0	0	0
2	-1.00	0	0	0	0	0	0	0	0
3	-1.00	0	0	0	0	0	0	0	0
4	-1.00	0	0	0	0	0	0	0	0
5	-1.00	-0.25	0	0	0	0	0	0	0
6	-1.00	-0.38	0	0	0	0	0	0	0
7	-1.00	-0.44	0	0	0	0	0	0	0
8	-1.00	-0.47	0	0	0	0	0	0	0
9	-1.00	-0.48	0	0	0	0	0	0	0
10	-1.00	-0.49	0	0	0	0	0	0	0
11	-1.00	-0.50	0	0	0	0	0	0	0
12	-1.00	-0.50	0	0	0	0	0	0	0
13	-1.00	-0.50	0	0	0	0	0	0	0
14	-1.00	-0.50	0	0	0	0	0	0	0
15	-1.00	-0.50	-0.12	0	0	0	0	0	0
16	-1.00	-0.53	-0.19	0	0	0	0	0	0
17	-1.00	-0.56	-0.23	0	0	0	0	0	0
18	-1.00	-0.59	-0.25	0	0	0	0	0	0
19	-1.00	-0.61	-0.27	0	0	0	0	0	0
20	-1.00	-0.62	-0.29	0	0	0	0	0	0
21	-1.00	-0.63	-0.30	0	0	0	0	0	0
22	-1.00	-0.64	-0.31	0	0	0	0	0	0
23	-1.00	-0.65	-0.31	0	0	0	0	0	0
24	-1.00	-0.65	-0.32	0	0	0	0	0	0

TABLE B-VII

TABLE B-VIII
 $\mu = 1.0$ $\alpha = 1.0$

k VALUES	n VALUES								
	0	1	2	3	4	5	6	7	8
0	-1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1	-1.00	0.50	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2	-1.00	0.25	0.88	1.00	1.00	1.00	1.00	1.00	1.00
3	-1.00	0.09	0.76	1.97	1.00	1.00	1.00	1.00	1.00
4	-1.00	0.09	0.64	0.95	0.99	1.00	1.00	1.00	1.00
5	-1.00	0.09	0.56	0.93	0.98	1.00	1.00	1.00	1.00
6	-1.00	0.09	0.51	0.92	0.97	1.00	1.00	1.00	1.00
7	-1.00	0.09	0.49	0.91	0.97	0.99	1.00	1.00	1.00
8	-1.00	0.09	0.47	0.90	0.96	0.99	1.00	1.00	1.00
9	-1.00	0.09	0.46	0.90	0.95	0.98	1.00	1.00	1.00
10	-1.00	0.09	0.46	0.90	0.94	0.98	0.99	1.00	1.00
11	-1.00	0.09	0.45	0.89	0.94	0.97	0.99	1.00	1.00
12	-1.00	0.09	0.45	0.89	0.93	0.97	0.99	1.00	1.00
13	-1.00	-0.14	0.45	0.89	0.93	0.97	0.99	0.99	1.00
14	-1.00	-0.21	0.41	0.89	0.93	0.96	0.98	0.99	1.00
15	-1.00	-0.25	0.37	0.88	0.92	0.96	0.98	0.99	1.00
16	-1.00	-0.28	0.34	0.87	0.92	0.96	0.98	0.99	1.00
17	-1.00	-0.30	0.32	0.86	0.91	0.95	0.98	0.99	1.00
18	-1.00	-0.32	0.30	0.86	0.91	0.95	0.97	0.99	1.00
19	-1.00	-0.34	0.28	0.85	0.90	0.95	0.97	0.99	1.00
20	-1.00	-0.35	0.27	0.84	0.90	0.94	0.97	0.98	1.00
21	-1.00	-0.36	0.26	0.84	0.90	0.94	0.97	0.98	1.00
22	-1.00	-0.36	0.25	0.84	0.89	0.94	0.96	0.98	1.00
23	-1.00	-0.37	0.24	0.83	0.89	0.93	0.96	0.98	1.00
24	-1.00	09.37	0.24	0.83	0.89	0.93	0.96	0.98	1.00

TABLE B-VIII

TABLE B-IX
 $\mu = 1.0$ $\alpha = 2.0$

k VALUES	n VALUES								
	0	1	2	3	4	5	6	7	8
0	-1.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00
1	-1.00	1.25	2.00	2.00	2.00	2.00	2.00	2.00	2.00
2	-1.00	0.88	1.81	2.00	2.00	2.00	2.00	2.00	2.00
3	-1.00	0.64	1.62	1.95	2.00	2.00	2.00	2.00	2.00
4	-1.00	0.48	1.46	1.92	1.98	2.00	2.00	2.00	2.00
5	-1.00	0.35	1.33	1.89	1.97	1.99	2.00	2.00	2.00
6	-1.00	0.26	1.22	1.87	1.96	1.99	1.99	2.00	2.00
7	-1.00	0.19	1.14	1.85	1.94	1.98	1.99	1.99	2.00
8	-1.00	0.13	1.08	1.84	1.93	1.97	1.99	1.99	2.00
9	-1.00	0.09	1.03	1.82	1.92	1.97	1.99	1.99	2.00
10	-1.00	0.05	1.00	1.81	1.91	1.96	1.98	1.99	2.00
11	-1.00	0.03	0.97	1.80	1.90	1.95	1.98	1.99	2.00
12	-1.00	0.03	0.94	1.79	1.89	1.95	1.98	1.99	2.00
13	-1.00	0.03	0.92	1.79	1.88	1.94	1.97	1.99	2.00
14	-1.00	0.03	0.91	1.78	1.87	1.93	1.97	1.98	2.00
15	-1.00	0.03	0.90	1.77	1.86	1.93	1.96	1.98	2.00
16	-1.00	0.03	0.89	1.77	1.86	1.92	1.96	1.98	2.00
17	-1.00	0.03	0.89	1.76	1.85	1.91	1.95	1.98	2.00
18	-1.00	0.03	0.89	1.76	1.84	1.91	1.95	1.97	2.00
19	-1.00	0.03	0.88	1.75	1.84	1.90	1.94	1.97	2.00
20	-1.00	0.03	0.88	1.75	1.83	1.90	1.94	1.97	2.00
21	-1.00	0.03	0.88	1.74	1.83	1.89	1.94	1.97	2.00
22	-1.00	0.03	0.88	1.74	1.82	1.89	1.93	1.96	2.00
23	-1.00	0.03	0.87	1.74	1.82	1.88	1.93	1.96	2.00
24	-1.00	0.03	0.87	1.73	1.81	1.88	1.92	1.96	2.00

TABLE B-IX

TABLE B-X
 $\mu = 1.0$ $\alpha = 3.0$

k VALUES	n VALUES								
	0	1	2	3	4	5	6	7	8
0	-1.00	3.00	3.00	3.00	3.00	3.00	3.00	3.00	3.00
1	-1.00	2.00	3.00	3.00	3.00	3.00	3.00	3.00	3.00
2	-1.00	1.50	2.75	3.00	3.00	3.00	3.00	3.00	3.00
3	-1.00	1.18	2.50	2.93	3.00	3.00	3.00	3.00	3.00
4	-1.00	0.97	2.28	2.90	2.98	3.00	3.00	3.00	3.00
5	-1.00	0.80	2.10	2.86	2.96	2.99	3.00	3.00	3.00
6	-1.00	0.68	1.97	2.83	2.94	2.99	2.99	3.00	3.00
7	-1.00	0.58	1.86	2.81	2.93	2.98	2.99	2.99	3.00
8	-1.00	0.51	1.78	2.79	2.91	2.97	2.99	2.99	3.00
9	-1.00	0.45	1.71	2.77	2.89	2.96	2.99	2.99	3.00
10	-1.00	0.40	1.66	2.75	2.88	2.95	2.98	2.99	3.00
11	-1.00	0.37	1.62	2.74	2.86	2.94	2.98	2.99	3.00
12	-1.00	0.34	1.58	2.73	2.85	2.93	2.97	2.99	3.00
13	-1.00	0.32	1.56	2.72	2.84	2.92	2.96	2.98	3.00
14	-1.00	0.30	1.54	2.71	2.83	2.91	2.96	2.98	3.00
15	-1.00	0.28	1.52	2.70	2.82	2.90	2.95	2.98	3.00
16	-1.00	0.27	1.50	2.67	2.87	2.89	2.98	2.97	3.00
17	-1.00	0.26	1.49	2.68	2.80	2.89	2.94	2.97	3.00
18	-1.00	0.26	1.48	2.67	2.79	2.88	2.93	2.97	3.00
19	-1.00	0.25	1.47	2.67	2.78	2.87	2.93	2.96	3.00
20	-1.00	0.24	1.46	2.66	2.78	2.86	2.92	2.96	3.00
21	-1.00	0.24	1.46	2.65	2.77	2.86	2.92	2.96	3.00
22	-1.00	0.23	1.45	2.65	2.76	2.85	2.91	2.95	3.00
23	-1.00	0.23	1.44	2.64	2.75	2.84	2.91	2.95	3.00
24	-1.00	0.23	1.44	2.64	2.75	2.84	2.90	2.94	3.00

TABLE B-X

TABLE B-XI

$\mu = 1.0$

$\sigma = 4.0$

k VALUES	n VALUES								
	0	1	2	3	4	5	6	7	8
0	-1.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00
1	-1.00	2.75	4.00	4.00	4.00	4.00	4.00	4.00	4.00
2	-1.00	2.12	3.68	4.00	4.00	4.00	4.00	4.00	4.00
3	-1.00	1.73	3.37	3.92	4.00	4.00	4.00	4.00	4.00
4	-1.00	1.46	3.10	3.88	3.98	4.00	4.00	4.00	4.00
5	-1.00	1.25	2.88	3.82	3.96	3.99	4.00	4.00	4.00
6	-1.00	1.09	2.71	3.79	3.93	3.98	3.99	4.00	4.00
7	-1.00	0.98	2.58	3.76	3.91	3.97	3.99	3.99	4.00
8	-1.00	0.88	2.47	3.73	3.89	3.96	3.99	3.99	4.00
9	-1.00	0.81	2.39	3.71	3.87	3.95	3.98	3.99	4.00
10	-1.00	0.75	2.32	3.69	3.88	3.94	3.88	2.99	4.00
11	-1.00	0.71	2.27	3.67	3.83	3.93	3.97	3.99	4.00
12	-1.00	0.67	2.23	3.66	3.82	3.91	3.96	3.99	4.00
13	-1.00	0.64	2.20	3.65	3.80	3.90	3.96	3.98	4.00
14	-1.00	0.62	2.27	3.63	3.79	3.89	3.95	3.98	4.00
15	-1.00	0.61	2.15	3.62	3.77	3.88	3.94	3.97	4.00
16	-1.00	0.59	2.13	3.61	3.76	3.87	3.93	3.97	4.00
17	-1.00	0.58	2.11	3.60	3.75	3.86	3.93	3.96	4.00
18	-1.00	0.57	2.10	3.59	3.74	3.85	3.92	3.96	4.00
19	-1.00	0.56	2.09	3.58	3.73	3.84	3.91	3.96	4.00
20	-1.00	0.55	2.08	3.57	3.72	3.83	3.90	3.95	4.00
21	-1.00	0.55	2.07	3.57	3.71	3.82	3.90	3.95	4.00
22	-1.00	0.54	2.06	3.56	3.70	3.81	3.89	3.94	4.00
23	-1.00	0.54	2.06	3.55	3.69	3.80	3.88	3.94	4.00
24	-1.00	0.53	2.05	3.55	3.69	3.80	3.88	3.93	4.00

TABLE B-XI

TABLE B-XII
 $\mu = 1.0$ $\alpha = 5.00$

k VALUES	n VALUES								
	0	1	2	3	4	5	6	7	8
0	-1.00	5.00	5.00	5.00	5.00	5.00	5.00	5.00	5.00
1	-1.00	3.50	5.00	5.00	5.00	5.00	5.00	5.00	5.00
2	-1.00	2.75	4.62	5.00	5.00	5.00	5.00	5.00	5.00
3	-1.00	2.28	4.25	4.90	5.00	5.00	5.00	5.00	5.00
4	-1.00	1.95	3.92	4.85	4.97	5.00	5.00	5.00	5.00
5	-1.00	1.70	3.66	4.79	4.95	4.99	5.00	5.00	5.00
6	-1.00	1.51	3.45	4.75	4.92	4.98	4.99	5.00	5.00
7	-1.00	1.37	3.29	4.71	4.89	4.97	4.99	4.99	5.00
8	-1.00	1.26	3.17	4.68	4.87	4.95	4.99	4.99	5.00
9	-1.00	1.17	3.07	4.65	4.84	4.94	4.98	4.99	5.00
10	-1.00	1.10	2.99	4.63	4.82	4.93	4.97	4.99	5.00
11	-1.00	1.05	2.93	4.61	4.80	4.91	4.97	4.99	5.00
12	-1.00	1.00	2.88	4.59	4.78	4.90	4.96	4.98	5.00
13	-1.00	0.97	2.84	4.58	4.76	4.88	4.95	4.98	5.00
14	-1.00	0.94	2.81	4.56	4.75	4.87	4.94	4.97	5.00
15	-1.00	0.93	2.78	4.55	4.73	4.86	4.93	4.97	5.00
16	-1.00	0.91	2.76	4.53	4.72	4.84	4.92	4.96	5.00
17	-1.00	0.89	2.74	4.52	4.70	4.83	4.91	4.96	5.00
18	-1.00	0.88	2.72	4.51	4.69	4.82	4.80	4.95	5.00
19	-1.00	0.87	2.71	4.50	4.68	4.81	4.89	4.95	5.00
20	-1.00	0.86	2.70	4.49	4.67	4.80	4.89	4.94	5.00
21	-1.00	0.86	2.69	4.48	4.65	4.79	4.88	4.94	5.00
22	-1.00	0.85	2.68	4.47	4.64	4.78	4.87	4.93	5.00
23	-1.00	0.85	2.67	4.46	4.63	4.77	4.86	4.92	5.00
24	-1.00	0.84	2.66	4.46	4.62	4.76	4.85	4.92	5.00

TABLE B-XII

TABLE B-XIII
 $\mu = 1.0$ $\alpha = 0$

k VALUES	n VALUES								
	0	1	2	3	4	5	6	7	8
0									
1	0.26	0.10	0	0	0	0	0	0	
2	0.75	0.23	0.04	0	0	0	0	0	
3	0.96	0.36	0.10	0.02	0	0	0	0	
4	1.15	0.49	0.17	0.04	0.01	0	0	0	
5	1.30	0.63	0.25	0.08	0.02	0	0	0	
6	1.40	0.75	0.33	0.12	0.04	0.01	0	0	
7	1.48	0.84	0.41	0.17	0.06	0.02	0	0	
8	1.50	0.93	0.49	0.22	0.09	0.03	0	0	
9	1.48	0.98	0.56	0.28	0.12	0.04	0.01	0	
10	1.40	1.01	0.61	0.33	0.15	0.06	0.02	0	
11	1.30	1.01	0.66	0.37	0.19	0.08	0.03	0	
12	1.15	0.98	0.68	0.41	0.22	0.10	0.04	0	
13	0.96	0.93	0.69	0.44	0.25	0.12	0.05	0	
14	0.75	0.85	0.69	0.47	0.28	0.14	0.06	0	
15	0.51	0.75	0.66	0.48	0.30	0.16	0.07	0	
16	0.26	0.62	0.62	0.48	0.32	0.18	0.08	0	
17	0	0.47	0.56	0.47	0.33	0.19	0.09	0	
18	-0.26	0.32	0.49	0.45	0.33	0.20	0.10	0	
19	-0.51	0.16	0.41	0.42	0.33	0.21	0.10	0	
20	-0.75	0	0.31	0.38	0.32	0.21	0.10	0	
21	-0.96	0	0.21	0.33	0.30	0.21	0.11	0	
22	-1.15	0	0.17	0.27	0.28	0.21	0.11	0	
23	-1.30	0	0.14	0.23	0.25	0.19	0.10	0	
24	-1.40	-0.34	0.12	0.20	0.22	0.18	0.10	0	

TABLE B-XIII

$\mu = 1.0$

$\alpha = 0$

k VALUES	n VALUES								
	0	1	2	3	4	5	6	7	8
25	-1.48	-0.58	0	0.18	0.20	0.16	0.09	0	
26	-1.50	-0.70	0	0.11	0.18	0.15	0.08	0	
27	-1.48	-0.74	0	0.09	0.14	0.13	0.08	0	
28	-1.40	-0.74	0	0.07	0.12	0.11	0.07	0	
29	-1.30	-0.71	0	0.06	0.10	0.10	0.06	0	
30	-1.15	-0.66	-0.21	0.05	0.08	0.08	0.05	0	
31	-0.96	-0.68	-0.29	0	0.07	0.07	0.04	0	
32	-0.75	-0.63	-0.33	0	0.04	0.06	0.04	0	
33	-0.51	-0.56	-0.32	0	0.03	0.04	0.03	0	
34	-0.26	-0.44	-0.29	0	0.02	0.03	0.02	0	
35	0	-0.31	-0.24	0	0.02	0.03	0.02	0	
36	0.26	-0.16	-0.17	0	0.01	0.02	0.01	0	
37	0.51	0	-0.10	0	0.01	0.02	0.01	0	
38	0.75	0	-0.02	0	0.01	0.01	0.01	0	
39	0.96	0	0	0	0.01	0.01	0.01	0	
40	1.15	0	0	0	0	0.01	0	0	
41	1.30	0.31	0	0	0	0.01	0	0	
42	1.40	0.58	0	0	0	0	0	0	
43	1.48	0.68	0	0	0	0	0	0	
44	1.50	0.73	0	0	0	0	0	0	
45	1.48	0.74	0	0	0	0	0	0	
46	1.40	0.74	0.21	0	0	0	0	0	
47	1.30	0.79	0.34	0	0	0	0	0	
48	1.15	0.81	0.39	0	0	0	0	0	
49	0.96	0.78	0.40	0	0	0	0	0	

TABLE B-XIII

$\mu = 1.0$

$\alpha = 0$

k VALUES	n VALUES								
	0	1	2	3	4	5	6	7	8
50	0.75	0.70	0.39	0	0	0	0	0	
51	0.57	0.60	0.36	0.05	0	0	0	0	
52	0.26	0.47	0.33	0.15	0.02	0	0	0	
53	0	0.33	0.31	0.17	0.07	0.01	0	0	
54	-0.26	0.19	0.26	0.19	0.09	0.03	0	0	
55	-0.57	0.04	0.20	0.18	0.10	0.04	0.01	0	
56	-0.75	0	0.13	0.16	0.10	0.05	0.02	0	
57	-0.96	0	0.09	0.13	0.11	0.06	0.03	0	
58	-1.15	0	0.07	0.10	0.10	0.07	0.03	0	
59	-1.30	-0.15	0.06	0.09	0.09	0.06	0.03	0	
60	-1.40	-0.53	0	0.07	0.08	0.06	0.03	0	
61	-1.48	-0.67	0	0.05	0.07	0.06	0.03	0	
62	-1.50	-0.72	0	0.04	0.05	0.05	0.03	0	
63	-1.48	-0.74	0	0.03	0.05	0.04	0.03	0	
64	-1.40	-0.74	0	0.02	0.04	0.04	0.02	0	
65	-1.30	-0.71	-0.29	0.02	0.03	0.03	0.02	0	
66	-1.15	-0.78	-0.33	0	0.03	0.03	0.01	0	
67	-0.96	-0.75	-0.38	0	0.02	0.02	0.01	0	
68	-0.75	-0.69	-0.38	0	0.01	0.02	0.01	0	
69	-0.51	-0.59	-0.35	0	0.01	0.01	0.01	0	
70	-0.26	-0.46	-0.30	0	0.01	0.01	0.01	0	
71	0	-0.32	-0.25	0	0.01	0.01	0.01	0	
72	0.26	-0.16	-0.18	0	0	0.01	0	0	
73	0.51	0	-0.10	0	0	0	0	0	
74	0.75	0	-0.02	0	0	0	0	0	

TABLE B-XIII

TABLE B-XIII
 $\mu = 1.0$ $\alpha = 0$

k VALUES	n VALUES								
	0	1	2	3	4	5	6	7	8
75	0.96	0	0	0	0	0	0	0	
76	1.15	0	0	0	0	0	0	0	
77	1.30	0.30	0	0	0	0	0	0	
78	1.40	0.48	0	0	0	0	0	0	
79	1.48	0.68	0	0	0	0	0	0	
80	1.50	0.73	0	0	0	0	0	0	
81	1.48	0.75	0	0	0	0	0	0	
82	1.40	0.74	0.21	0	0	0	0	0	
83	1.30	0.79	0.34	0	0	0	0	0	
84	1.15	0.81	0.39	0	0	0	0	0	
85	0.96	0.78	0.40	0	0	0	0	0	
86	0.75	0.70	0.39	0	0	0	0	0	
87	0.51	0.60	0.36	0	0	0	0	0	
88	0.26	0.47	0.31	0	0	0	0	0	
89	0	0.32	0.25	0	0	0	0	0	
90	-0.26	0.16	0.18	0.08	0	0	0	0	
91	-0.51	0	0.13	0.09	0.03	0	0	0	
92	-0.75	0	0.06	0.08	0.04	0.01	0	0	
93	-0.96	0	0.05	0.06	0.05	0.02	0	0	
94	-1.15	0	0.03	0.05	0.04	0.03	0.01	0	
95	-1.30	-0.24	0.03	0.04	0.04	0.02	0.01	0	
96	-1.40	-0.56	0	0.03	0.03	0.02	0.01	0	
97	-1.48	-0.67	0	0.02	0.03	0.02	0.01	0	
98	-1.50	-0.73	0	0.02	0.02	0.02	0.01	0	
99	-1.48	-0.75	0	0.01	0.02	0.02	0.01	0	

TABLE B-XIII

 $\mu = 1.0$ $\alpha = 0$

k VALUES	n VALUES								
	0	1	2	3	4	5	6	7	8
100	-1.40	-0.74	-0.12	0.01	0.02	0.02	0.01	0	
101	-1.30	-0.76	-0.32	0	0.01	0.01	0.01	0	
102	-1.15	-0.80	-0.37	0	0.01	0.01	0.01	0	
103	-0.96	-0.77	-0.39	0	0.01	0.01	0.01	0	
104	-0.75	-0.70	-0.38	0	0	0.01	0	0	
105	-0.51	-0.59	-0.36	0	0	0	0	0	
106	-0.26	-0.47	-0.31	0	0	0	0	0	
107	0	-0.32	-0.25	0	0	0	0	0	
108	0.26	-0.16	-0.18	0	0	0	0	0	

k VALUES	n VALUES								
	0	1	2	3	4	5	6	7	8
0									
1	1.86	1.17	1.00	1.00	1.00	1.00	1.00	1.00	
2	2.25	1.38	1.07	1.00	1.00	1.00	1.00	1.00	
3	2.60	1.60	1.16	1.03	1.00	1.00	1.00	1.00	
4	2.92	1.83	1.29	1.07	1.01	1.00	1.00	1.00	
5	3.17	2.05	1.42	1.13	1.03	1.00	1.00	1.00	
6	3.34	2.24	1.56	1.21	1.06	1.01	1.00	1.00	
7	3.46	2.41	1.69	1.29	1.10	1.03	1.01	1.00	
8	3.50	2.54	1.82	1.37	1.15	1.05	1.01	1.00	
9	3.46	2.64	1.93	1.46	1.20	1.07	1.02	1.00	
10	3.34	2.68	2.02	1.54	1.25	1.10	1.03	1.00	
11	3.17	2.68	2.10	1.62	1.31	1.13	1.05	1.00	
12	2.92	2.64	2.14	1.69	1.36	1.17	1.06	1.00	
13	2.61	2.55	2.16	1.74	1.41	1.20	1.08	1.00	
14	2.25	2.42	2.15	1.78	1.46	1.24	1.10	1.00	
15	1.86	2.24	2.11	1.80	1.50	1.27	1.12	1.00	
16	1.43	2.03	2.04	1.80	1.53	1.30	1.13	1.00	
17	1.00	1.80	1.94	1.79	1.55	1.32	1.15	1.00	
18	0.57	1.54	1.82	1.75	1.55	1.34	1.16	1.00	
19	0.14	1.26	1.68	1.70	1.55	1.35	1.17	1.00	
20	-0.25	0.98	1.52	1.63	1.53	1.36	1.17	1.00	
21	-0.61	0.70	1.35	1.55	1.50	1.35	1.18	1.00	
22	-0.92	0.44	1.17	1.45	1.46	1.34	1.18	1.00	
23	-1.17	0.19	0.99	1.34	1.41	1.32	1.17	1.00	
24	-1.34	0	0.81	1.23	1.35	1.30	1.16	1.00	

TABLE B-XIV
 $\mu = 1.0$ $\alpha = 1.0$

k VALUES	n VALUES								
	0	1	2	3	4	5	6	7	8
25	-1.46	0	0.65	1.11	1.28	1.26	1.15	1.00	
26	-1.50	0	0.57	0.99	1.21	1.23	1.14	1.00	
27	-1.46	0	0.51	0.91	1.13	1.18	1.12	1.00	
28	-1.34	-0.32	0.47	0.84	1.06	1.13	1.10	1.00	
29	-1.17	-0.41	0.30	0.78	1.00	1.09	1.07	1.00	
30	-0.92	-0.43	0.21	0.68	0.95	1.05	1.05	1.00	
31	-0.61	-0.37	0.14	0.60	0.88	1.00	1.03	1.00	
32	-0.25	-0.26	0.12	0.53	0.82	0.97	1.00	1.00	
33	-0.14	-0.10	0.13	0.48	0.76	0.92	0.99	1.00	
34	0.57	0	0.18	0.45	0.71	0.88	0.97	1.00	
35	1.00	0	0.22	0.45	0.68	0.85	0.95	1.00	
36	1.43	0	0.22	0.45	0.65	0.82	0.93	1.00	
37	1.86	0.54	0.22	0.44	0.64	0.80	0.91	1.00	
38	2.25	0.94	0.43	0.43	0.62	0.78	0.90	1.00	
39	2.61	1.26	0.64	0.51	0.61	0.77	0.89	1.00	
40	2.92	1.55	0.84	0.60	0.63	0.75	0.88	1.00	
41	3.17	1.81	1.03	0.71	0.67	0.76	0.88	1.00	
42	3.34	2.04	1.21	0.82	0.72	0.77	0.88	1.00	
43	3.46	2.23	1.39	0.94	0.78	0.79	0.88	1.00	
44	3.50	2.39	1.54	1.05	0.85	0.82	0.89	1.00	
45	3.46	2.49	1.68	1.17	0.92	0.86	0.91	1.00	
46	3.34	2.56	1.80	1.28	1.00	0.90	0.93	1.00	
47	3.17	2.57	1.89	1.37	1.07	0.95	0.95	1.00	
48	2.92	2.54	1.96	1.46	1.14	1.00	0.97	1.00	
49	2.61	2.46	1.99	1.53	1.21	1.04	0.99	1.00	

k VALUES	n VALUES								
	0	1	2	3	4	5	6	7	8
50	2.25	2.33	1.99	1.59	1.27	1.09	1.02	1.00	
51	1.86	2.16	1.97	1.62	1.33	1.13	1.04	1.00	
52	1.43	1.96	1.91	1.64	1.37	1.17	1.06	1.00	
53	1.00	1.73	1.82	1.64	1.40	1.21	1.08	1.00	
54	0.56	1.47	1.71	1.62	1.42	1.23	1.10	1.00	
55	0.15	1.21	1.48	1.48	1.42	1.25	1.11	1.00	
56	-0.25	0.93	1.43	1.52	1.42	1.27	1.12	1.00	
57	-0.61	0.66	1.26	1.44	1.40	1.27	1.13	1.00	
58	-0.92	0.39	1.09	1.35	1.36	1.26	1.13	1.00	
59	-0.17	0.15	0.92	1.25	1.32	1.25	1.13	1.00	
60	-1.34	0	0.74	1.15	1.27	1.23	1.13	1.00	
61	-1.46	0	0.61	1.03	1.20	1.20	1.12	1.00	
62	-1.50	0	0.53	0.93	1.14	1.17	1.10	1.00	
63	-1.46	-0.94	0.48	0.85	1.07	1.13	1.09	1.00	
64	-1.34	-0.40	0.42	0.79	1.01	1.09	1.07	1.00	
65	-1.17	-0.45	0.24	0.73	0.95	1.05	1.05	1.00	
66	-0.92	-0.46	0.16	0.62	0.90	1.01	1.03	1.00	
67	-0.61	-0.39	0.10	0.55	0.83	0.97	1.01	1.00	
68	-0.25	-0.28	0.08	0.48	0.78	0.93	0.99	1.00	
69	0.14	-0.12	0.10	0.44	0.72	0.89	0.97	1.00	
70	0.57	0	0.14	0.41	0.68	0.86	0.95	1.00	
71	1.00	0	0.20	0.41	0.64	0.82	0.93	1.00	
72	1.43	0	0.20	0.42	0.62	0.80	0.92	1.00	
73	1.86	0.49	0.21	0.21	0.61	0.77	0.90	1.00	
74	2.25	0.92	0.40	0.41	0.60	0.76	0.89	1.00	

TABLE B-XIV
 $\mu = 1.0$ $\alpha = 1.0$

k VALUES	n VALUES								
	0	1	2	3	4	5	6	7	8
75	2.61	1.25	0.61	0.48	0.59	0.75	0.88	1.00	
76	2.92	1.54	0.81	0.58	0.61	0.74	0.88	1.00	
77	3.17	1.80	1.01	0.68	0.65	0.74	0.87	1.00	
78	3.34	2.03	1.20	0.80	0.70	0.76	0.87	1.00	
79	3.46	2.22	1.37	0.92	0.76	0.78	0.88	1.00	
80	3.50	2.38	1.53	1.04	0.83	0.81	0.89	1.00	
81	3.46	2.49	1.67	1.15	0.91	0.85	0.90	1.00	
82	3.34	2.55	1.79	1.26	0.98	0.89	0.92	1.00	
83	3.17	2.56	1.88	1.36	1.06	0.94	0.94	1.00	
84	2.92	2.53	1.95	1.45	1.13	0.99	0.96	1.00	
85	3.61	2.45	1.98	1.52	1.20	1.04	0.99	1.00	
86	2.25	2.33	1.98	1.48	1.26	1.08	1.01	1.00	
87	1.86	2.16	1.96	1.61	1.32	1.13	1.04	1.00	
88	1.43	1.96	1.90	1.63	1.36	1.17	1.06	1.00	
89	1.00	1.73	1.82	1.63	1.39	1.20	1.08	1.00	
90	0.57	1.47	1.71	1.61	1.41	1.23	1.10	1.00	
91	0.15	1.20	1.57	1.57	1.42	1.25	1.11	1.00	
92	-0.25	0.93	1.42	1.51	1.41	1.26	1.12	1.00	
93	-0.61	0.65	1.26	1.44	1.39	1.26	1.13	1.00	
94	-0.92	0.39	1.09	1.35	1.36	1.26	1.13	1.00	
95	-1.17	0.15	0.91	1.25	1.31	1.25	1.13	1.00	
96	-1.34	0	0.74	1.14	1.26	1.23	1.13	1.00	
97	-1.46	0	0.60	1.03	1.20	1.20	1.12	1.00	
98	-1.50	0	0.53	0.93	1.13	1.17	1.10	1.00	
99	-1.46	-0.04	0.48	0.85	1.06	1.13	1.09	1.00	

TABLE B-XIV

 $\mu = 1.0$ $\alpha = 1.0$

k VALUES	n VALUES								
	0	1	2	3	4	5	6	7	8
100	-1.34	-4.02	0.42	0.79	1.00	1.09	1.07	1.00	
101	-1.17	-0.45	0.24	0.73	0.95	1.05	1.05	1.00	
102	-0.92	-0.46	0.16	0.62	0.90	1.01	1.03	1.00	
103	-0.61	-0.40	0.10	0.55	0.83	0.97	1.01	1.00	
104	-0.25	-0.28	0.08	0.48	0.77	0.93	0.99	1.00	
105	0.15	-0.12	0.09	0.44	0.72	0.89	0.97	1.00	
106	0.57	0	0.14	0.41	0.68	0.85	0.95	1.00	
107	1.00	0	0.19	0.41	0.64	0.82	0.93	1.00	
108	1.43	0	0.20	0.42	0.62	0.79	0.92	1.00	

APPENDIX C

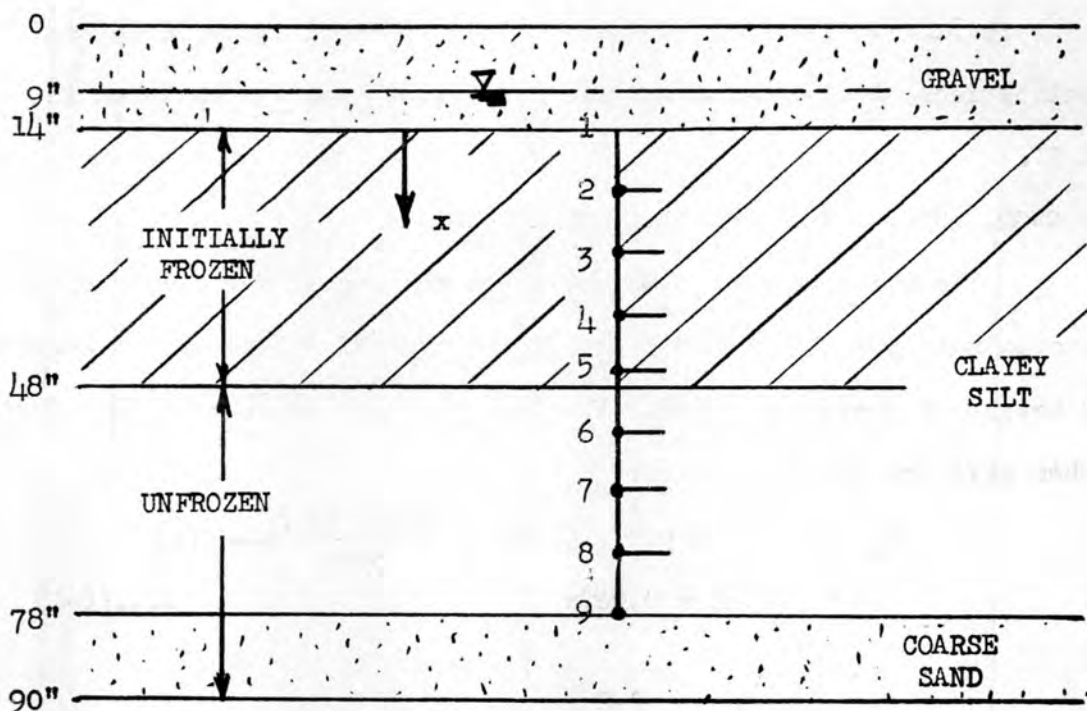
NUMERICAL SOLUTION OF A THAW-CONSOLIDATION PROBLEM

It is intended to show by the following example a general computational procedure for solving a thaw-consolidation problem numerically. The physical nature of the problem and its mathematical formulation are discussed in Part IV. For the sake of clarity in solution technique, conditions for this example have been simplified. They may not represent a realistic case.

C-01. CONDITIONS AND ASSUMPTIONS.

Assume for the purpose of this example that the following conditions and assumptions hold:

(1.) The soil profile applying to this problem is the same as that shown in Figure 11. The sketch given below shows the profile at a time when the clayey silt just begins to thaw. This situation is denoted as zero time.



(2.) The clayey silt has the following properties in the unfrozen consolidated state:

$$\begin{aligned}\gamma_t &= 120 \text{ lb. per cu. ft.} \\ e_1 &= 0.9 \text{ at } x = 0 \\ k &= 3.5 \times 10^{-8} \text{ ft. per sec.} \\ a_v &= 0.1 \text{ sq. ft. per ton} \\ c_v &= 2.1 \times 10^{-5} \text{ sq. ft. per sec.}\end{aligned}$$

It is assumed that these properties, with the exception of e_1 , do not change appreciably after ice segregation and during subsequent consolidation.

(3.) Ground surface has heaved a total of 3.6", 1.6" of which can be accounted for by volume expansion as the soil moisture freezes. The remaining 2" results from lense growth in the clayey silt which is assumed to be distributed uniformly with depth in the frozen zone.

(4.) The average rate of thawing is linear at 1/16" per hour and the soil follows a path similar to line BC'A of Figure 10 during consolidation. A discussion of this simplified case is presented in Part IV.

C-02. SOIL CONDITIONS PRIOR TO THAWING. -

The intergranular pressure p_1 at the top of the clayey silt is approximately 0.055 tons per sq. ft. if typical values of the effective unit weight of gravel are used. The intergranular pressure at any depth x is then given by the expression:

$$\begin{aligned}p_x &= p_1 + \Delta p = 0.055 + \frac{120 - 62.4}{2000} (x) \\ &= 0.055 + 0.029x \quad \dots(C-1)\end{aligned}$$

From condition (2.), it follows that the void ratio at any depth x before freezing is:

$$\begin{aligned} e_{Ax} &= e_1 + a_v \Delta p \\ &= 0.900 + 0.0029x \end{aligned} \quad \dots (C-2)$$

This void ratio at any depth is indicated by point A in Figure 10.

From condition (3.), it may be shown that the void ratio increase Δe as a result of 2-inch, ice lense growth is:

$$\Delta e = \frac{2(1+e)}{34} = 0.112 \quad \dots (C-3)$$

The void ratio within the frozen portion of the clayey silt immediately after it thaws is therefore:

$$e_{Bx} = \Delta e + e_{Ax} = 1.012 + 0.0029x \quad \dots (C-4)$$

which is equivalent to point B in Figure 10. The void ratio corresponding to point C' in the same figure is equal to 0.9055 for all depths.

For the numerical solution, the clayey silt will be considered to consist of 8 layers, each 8 inches thick, a subdivision identical to that made in Figure 11 for the hydraulic model. According to the condition that the soil is represented by point B in Figure 10 immediately after thawing, the initial hydrostatic excess water pressure u_x is numerically equal to p_x given by Equation (C-1). The excess pressure remains equal to this value until the void ratio at any point decreases to 0.9055 at which time the gradual transfer of overburden load from the pore water to the grain structure commences.

The following table applies then to the conditions of this problem:

Point n	Depth inches	x ft.	initial u tons per sq. ft.	e x before freezing	e x immediately after thawing
1	14	0	0(0.055)	0.9000	1.0120
2	22	0.67	0.074	0.9019	1.0139
3	30	1.33	0.094	0.9039	1.0159
4	38	2.0	0.113	0.9058	1.0178
5	46	2.67	0.132	0.9077	1.0197
6	54	3.33	0	0.9097	0.9097
7	62	4.0	0	0.9116	0.9116
8	70	4.67	0	0.9135	0.9135
9	78	5.33	0	0.9154	0.9154

C-03. NUMERICAL SOLUTION. -

It has been demonstrated in Section 2-01-d that the finite difference approximation of the diffusion equation for heat flow may be written as Equation (2 - 17c) for the one dimensional case. The corresponding equation for consolidation becomes:

$$u_{n,k+1} = \beta (u_{n-1,k} + u_{n+1,k}) + (1-2\beta)u_{n,k} \dots (C-5)$$

If we select a value of β equal to $1/3$ for this solution then Equation (C-5) becomes:

$$u_{n,k+1} = \frac{1}{3} (u_{n-1,k} + u_{n,k} + u_{n+1,k}) \dots (C-6)$$

Since:

$$\beta = \frac{c_v \Delta t}{\Delta s^2} \dots (C-7)$$

and since we have selected $\Delta s = 8''$ then:

$$\Delta t = \frac{\beta \Delta s^2}{c_v} = \frac{\frac{1}{2} \left(\frac{8}{12} \right)^2}{2.1 \times 10^{-5}}$$

$$= 7100 \text{ sec.}$$

Say $\Delta t = 2 \text{ hrs.}$ (time interval between steps)

Equation (C-6) applies while the soil is consolidating along a path given by line C'A of Figure 10. Along line BC' the excess pressure remains constant while the void ratio decreases.

The finite difference approximation for void ratio change may be written:

$$K_{e_{n,k+1}} = K_{e_{n,k}} + \beta (u_{n+1,k} + u_{n-1,k} - 2u_{n,k}) \dots (C-8)$$

which is comparable to Equation (2-18) of Section 2-01-f-(3).

Equation (C-8) may be rearranged:

$$K_{e_{n,k+1}} = K_{e_{n,k}} + \beta (u_{n-1,k} - u_{n,k}) - \beta (u_{n,k} - u_{n+1,k}) \dots (C-9)$$

and when $\beta = 1/3$ it may be written:

$$K \Delta e = \frac{1}{3} (\Delta u_1 - \Delta u_2) \dots (C-10)$$

We must now find K before proceeding with the numerical solution.

In each two-hour period the volume of water ΔQ flowing from an 8-inch layer is given by Darcy's Law:

$$\begin{aligned} \Delta Q &= Q_{in} - Q_{out} = -k \frac{\Delta H_2}{\Delta s} 7200 + k \frac{\Delta H_1}{\Delta s} 7200 \\ &= \frac{7200k}{\Delta s} (\Delta H_1 - \Delta H_2) \\ &= \frac{7200k}{\Delta s} \frac{1}{62.4} (\Delta u_1 - \Delta u_2) \\ &= 6 \times 10^{-6} (\Delta u_1 - \Delta u_2) \text{ cu. ft.} \end{aligned}$$

since:

$$\begin{aligned}\Delta e &= \frac{\Delta v_v}{v_s} = \frac{\Delta Q}{V(1-n)} \\ &= \frac{6 \times 10^{-6} (\Delta u_1 - \Delta u_2)}{\frac{8}{12} (1 - \frac{0.9}{1.9})} \\ &= 17.3 \times 10^{-6} (\Delta u_1 - \Delta u_2)\end{aligned}$$

then it follows from Equation (C-10) that:

$$K = \frac{\frac{1}{3}}{17.3 \times 10^{-6} \times 2000} = 9.6 \text{ tons per sq. ft.}$$

The numerical solution will be carried out in a tabular form similar to that shown in Table II. Notation for this example is as follows:

	k	k+1
n-1	$u_{n-1,k}$	$Ke_{n-1,k}$
	$u_{n-1,k} - u_{n,k}$	$\beta(u_{n-1,k} - u_{n,k})$
n	$u_{n,k}$	$Ke_{n,k}$
	$u_{n,k} - u_{n+1,k}$	$\beta(u_{n,k} - u_{n+1,k})$
n+1	$u_{n+1,k}$	$Ke_{n+1,k}$

Equations (C-6) and (C-9) govern the numerical process which is now presented.

NUMERICAL SOLUTION OF A
THAW - CONSOLIDATION PROBLEM

$\frac{k}{t}$ $\frac{X_f}{X_f}$ n	0 0 0	1 2 hrs. 1/8 in.	2 4 hrs. 1/4 in.	63 126 hrs. 7-7/8 ins.	64 128 hrs. 8 ins.	65 130 hrs. 8-1/8 ins.	66 132 hrs. 8-1/4 ins.	105 210 hrs. 13-1/8 ins.	106 212 hrs. 13-1/4 ins.	107 214 hrs. 13-3/8 ins.	108 216 hrs. 13-1/2 ins.
1	0.055	9.7152	0 8.6400	0 8.6400	0 8.6400	0 8.6400	0 8.6400	0 8.6400	0 8.6400	0 8.6400	0 8.6400
2	0.074	9.7334	0.074 9.7334	0.074 9.7334	0.074 9.7334	0.074 9.7087	0.074 9.6840	0.074 8.7087	0.074 8.6928	0.0247 8.6681	0.0082 8.6599
3	0.094	9.7526	0.094 9.7526	0.094 9.7526	Flow commences from Point 2 toward surface	9.7334/(-0.0247) = 7.7087 follows from Equation (C-9)			Void ratio at Point 2 has reached 0.9055 and excess pres- sure at Point 2 begins to dissi- pate according to Equation (C-6)		It is apparent that the excess pressure at Point 2 will be essentially dis- sipated when the thawing pene- trates to 16 ins.
4	0.113	9.7709	0.113 9.7709								
5	0.132	9.7891	0.132 9.7891								
6	0		0								
7	0		0								
8	0		0								
9	0		0								
REMARKS	No flow occurs until the depth of thawing X_f penetrates 8 inches										

$\frac{k}{t}$ $\frac{X_f}{X_f}$ n	127 254 hrs. 15-7/8 ins.	128 256 hrs. 16 ins.	129 258 hrs. 16-1/8 ins.	130 260 hrs. 16-1/4 ins.	131 262 hrs. 16-3/8 ins.	132 264 hrs. 16-1/2 ins.	133 266 hrs. 16-5/8 ins.
1	0 8.6400	0 8.6400	0 8.6400	0 8.6400	0 8.6400	0 8.6400	0 8.6400
2	0 8.6582	0 8.6582	0.0313 8.6582	0.0418 8.6687	0.0453 8.6722	0.0464 8.6833	0.0464 8.6837
3	0.094 9.7526	0.094 9.7526	0.094 9.7213	0.094 9.7004	0.094 9.6830	0.094 9.6668	
		Flow commences from Point 3 toward surface				Note that the soil at Point 2 swells as water from Point 3 begins to flow upward.	The solution pro- ceeds in like manner as each new layer thaws and starts to consolidate.